

Multiplicity of solutions of a zero mass nonlinear equation on a Riemannian manifold

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1 Introduction

In this paper we are interested in the relation between the number of solutions of a nonlinear equation on a Riemannian manifold and the topology of the manifold itself.

Let (M, g) be a compact, connected, orientable, boundaryless Riemannian manifold of class C^∞ with Riemannian metric g . Let $\dim(M) = n \geq 3$.

We consider the problem

$$-\epsilon^2 \Delta u = f'(u) \quad (1.1)$$

with $u \in H_1^2(M)$.

As it has been pointed out in [10] problem (1.1) admits solutions on \mathbb{R}^n if $f'(0) < 0$, while there are no solutions if $f'(0) > 0$. The limiting case $f'(0) = 0$, i.e. the “zero mass” case, depends on the structure of f . Berestycki and Lions proved the existence of ground state solutions if $f(u)$ behaves as $|u|^p$ for u large and $|u|^q$ for u small, with p and q respectively super and sub-critical. In [9] they proved also the existence of infinitely many bound state solutions.

Problem (1.1) has been studied also in [8], where existence and non existence results have been given on an exterior domain in \mathbb{R}^n .

The problem of the multiplicity of solutions of a nonlinear elliptic equation on a Riemannian manifold has been studied in [3], where the authors consider an equation with sub-critical growth.

The effect of the domain shape on the number of positive solutions of some semilinear elliptic problems has been widely studied. Here we only mention [1], [11], [5], [6] and [4].

Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an even function such that:

$$(f1) \quad 0 < \mu f(s) \leq f'(s)s < f''(s)s^2 \text{ for any } s \neq 0 \text{ and for some } \mu > 2;$$

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(f2) $f(0) = f'(0) = f''(0) = 0$ and there exist positive constants c_0, c_1, p, q with $2 < p < 2^* < q$ such that

$$f(s) \geq \begin{cases} c_0 |s|^p & \text{for } |s| \geq 1 \\ c_0 |s|^q & \text{for } |s| \leq 1 \end{cases} \quad (1.2)$$

$$f''(s) \geq \begin{cases} c_1 |s|^{p-2} & \text{for } |s| \geq 1 \\ c_1 |s|^{q-2} & \text{for } |s| \leq 1 \end{cases} \quad (1.3)$$

We denote by $\text{cat}(M)$ the Ljusternik-Schnirelmann category of M and by $\mathcal{P}_t(M)$ the Poincaré polynomial of M .

Our main results are the following:

Theorem 1.1. *For $\epsilon > 0$ sufficiently small, equation (1.1) has at least $\text{cat}(M) + 1$ solutions in $H_1^2(M)$.*

Theorem 1.2. *If for $\epsilon > 0$ sufficiently small the solutions of equation (1.1) are non-degenerate, then there are at least $2\mathcal{P}_1(M) - 1$ solutions.*

2 Notation and preliminary results

We denote by $B(0, R)$ the ball in \mathbb{R}^n of centre 0 and radius R and by $B_g(x, R)$ the ball in M of centre x and radius R .

We define a smooth real function χ_R on \mathbb{R}^+ such that

$$\chi_R(t) = \begin{cases} 1 & \text{if } 0 \leq t \leq \frac{R}{2} \\ 0 & \text{if } t \geq R \end{cases} \quad (2.1)$$

and $|\chi_R'(t)| \leq \frac{\chi_0}{R}$, with χ_0 positive constant.

We recall some definitions and results about compact connected Riemannian manifolds of class C^∞ (see for example [13]).

Remark 2.1. *On the tangent bundle TM of M the exponential map $\exp : TM \rightarrow M$ is defined. This map has the following properties:*

- (i) \exp is of class C^∞ ;
- (ii) there exists a constant $R > 0$ such that

$$\exp_x|_{B(0, R)} : B(0, R) \rightarrow B_g(x, R)$$

is a diffeomorphism for all $x \in M$.

It is possible to choose an atlas \mathcal{C} on M , whose charts are given by the exponential map (normal coordinates). We denote by $\{\psi_C\}_{C \in \mathcal{C}}$ a partition of unity subordinate to the atlas \mathcal{C} . Let g_{x_0} be the Riemannian metric in the normal coordinates of the map \exp_{x_0} .

For any $u \in H_1^2(M)$ we have that:

$$\begin{aligned} \int_M |\nabla u(x)|_g^2 d\mu_g &= \sum_{C \in \mathcal{C}} \int_C \psi_C(x) |\nabla u(x)|_g^2 d\mu_g \\ &= \sum_{C \in \mathcal{C}} \int_{B(0,R)} \psi_C(\exp_{x_C}(z)) g_{x_C}^{ij}(z) \frac{\partial u(\exp_{x_C}(z))}{\partial z_i} \frac{\partial u(\exp_{x_C}(z))}{\partial z_j} |g_{x_C}(z)|^{\frac{1}{2}} dz, \end{aligned}$$

where Einstein notation is adopted, that is

$$g^{ij} z_i z_j = \sum_{i,j=1}^n g^{ij} z_i z_j,$$

$(g_{x_0}^{ij}(z))$ is the inverse matrix of $g_{x_0}(z)$ and $|g_{x_0}(z)| = \det(g_{x_0}(z))$. In particular we have that $g_{x_0}(0) = \text{Id}$. A similar relation holds for the integration of $|u(x)|^p$. For convenience we will also write for all $x_0 \in M$ and $z, \xi \in T_{x_0}M$

$$|\xi|_{g_{x_0}(z)}^2 = g_{x_0}^{ij}(z) \xi_i \xi_j. \quad (2.2)$$

Remark 2.2. Since M is compact, there are two strictly positive constants h and H such that for all $x \in M$ and all $z \in T_x M$

$$h|z|^2 \leq g_x(z, z) \leq H|z|^2,$$

where $|\cdot|$ is the standard metric in \mathbb{R}^n . Hence there holds

$$h^n \leq |g_x(z)| \leq H^n.$$

We are going to find the solutions of (1.1) as critical points of the functional $J_\epsilon : H_1^2(M) \rightarrow \mathbb{R}$, defined by

$$J_\epsilon(u) = \frac{\epsilon^2}{2\epsilon^n} \int_M |\nabla u(x)|_g^2 d\mu_g - \frac{1}{\epsilon^n} \int_M f(u(x)) d\mu_g, \quad (2.3)$$

constrained on the Nehari manifold

$$\mathcal{N}_\epsilon = \left\{ u \in H_1^2(M) \mid u \neq 0 \text{ and } \int_M \epsilon^2 |\nabla u|_g^2 d\mu_g = \int_M f'(u)u d\mu_g \right\}. \quad (2.4)$$

Let $\mathcal{D}^{1,2}(\mathbb{R}^n)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|v\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |\nabla v(z)|^2 dz.$$

We consider also the following functional $J : \mathcal{D}^{1,2}(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by

$$J(v) := \int_{\mathbb{R}^n} \left(\frac{1}{2} |\nabla v(x)|^2 - f(v(x)) \right) dx \quad (2.5)$$

and the associated Nehari manifold

$$\mathcal{N} = \left\{ v \in \mathcal{D}^{1,2}(\mathbb{R}^n) \mid v \neq 0 \text{ and } \int_{\mathbb{R}^n} |\nabla v(x)|^2 dx = \int_{\mathbb{R}^n} f'(u)u dx \right\}. \quad (2.6)$$

The functionals J_ϵ and J are C^2 respectively on $H_1^2(M)$ and on $\mathcal{D}^{1,2}(\mathbb{R}^n)$. In fact, we have:

Lemma 2.3. (i) *The functional $F_{\epsilon,M} : L^p(M) \rightarrow \mathbb{R}$, defined by*

$$F_{\epsilon,M}(u) := \frac{1}{\epsilon^n} \int_M f(u(x)) d\mu_g \quad (2.7)$$

is of class C^2 and

$$\begin{aligned} F'_{\epsilon,M}(u_0)u_1 &= \frac{1}{\epsilon^n} \int_M f'(u_0(x))u_1(x) d\mu_g \\ F''_{\epsilon,M}(u_0)u_1u_2 &= \frac{1}{\epsilon^n} \int_M f''(u_0(x))u_1(x)u_2(x) d\mu_g \end{aligned}$$

(ii) *The functional $F : L^{2^*}(\mathbb{R}^n) \rightarrow \mathbb{R}$ defined by*

$$F(v) := \int_{\mathbb{R}^n} f(v(z)) dz \quad (2.8)$$

is of class C^2 and

$$\begin{aligned} F'(v_0)v_1 &= \int_{\mathbb{R}^n} f'(v_0(z))v_1(z) dz \\ F''(v_0)v_1v_2 &= \int_{\mathbb{R}^n} f''(v_0(z))v_1(z)v_2(z) dz \end{aligned}$$

The proof of this lemma is analogous to the proof of Lemma 2.7 in [8].

We also have the following lemma:

Lemma 2.4. *The functionals $\tilde{F}_{\epsilon,M} : L^p(M) \rightarrow \mathbb{R}$, defined by*

$$\tilde{F}_{\epsilon,M}(u) := \frac{1}{\epsilon^n} \int_M \left[\frac{1}{2} f'(u(x))u(x) - f(u(x)) \right] d\mu_g \quad (2.9)$$

and $\tilde{F}_\Omega : L^{2^}(\Omega) \rightarrow \mathbb{R}$ defined by*

$$\tilde{F}_\Omega(v) := \int_\Omega \left[\frac{1}{2} f'(v(z))v(z) - f(v(z)) \right] dz \quad (2.10)$$

are strongly continuous.

We write

$$m(J) := \inf\{J(v) \mid v \in \mathcal{N}\}. \quad (2.11)$$

There exists a positive spherically symmetric and decreasing with $|z|$ solution $U \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ of

$$-\Delta U = f'(U) \quad \text{in } \mathbb{R}^n, \quad (2.12)$$

such that $J(U) = m(J)$ (see [10] and [8]).

The function $U_\epsilon(z) = U\left(\frac{z}{\epsilon}\right)$ is solution of

$$-\epsilon^2 \Delta U_\epsilon = f(U_\epsilon).$$

For any $\delta > 0$ we consider the subset of \mathcal{N}_ϵ

$$\Sigma_{\epsilon,\delta} := \{u \in \mathcal{N}_\epsilon \mid J_\epsilon(u) < m(J) + \delta\}. \quad (2.13)$$

We recall now the definition of Palais-Smale condition:

Definition 2.5. *Let J be a C^1 functional on a Banach space X . A sequence $\{u_m\}$ in X is a Palais-Smale sequence for J if $|J(u_m)| \leq c$, uniformly in m , while $J'(u_m) \rightarrow 0$ strongly, as $m \rightarrow \infty$. We say that J satisfies the Palais-Smale condition ((PS) condition) if any Palais-Smale sequence has a convergent subsequence.*

3 Ideas of the proof for the category theory result

We recall the definition of Ljusternik-Schnirelmann category (see [14]).

Definition 3.1. *Let M be a topological space and consider a closed subset $A \subset M$. We say that A has category k relative to M ($\text{cat}_M(A) = k$) if A is covered by k closed sets A_j , $1 \leq j \leq k$, which are contractible in M and if k is minimal with this property. If no such finite covering exists, we let $\text{cat}_M(A) = \infty$. If $A = M$, we write $\text{cat}_M(M) = \text{cat}(M)$.*

Remark 3.2. *Let M_1 and M_2 be topological spaces. If $g_1 : M_1 \rightarrow M_2$ and $g_2 : M_2 \rightarrow M_1$ are continuous operators such that $g_2 \circ g_1$ is homotopic to the identity on M_1 , then $\text{cat}(M_1) \leq \text{cat}(M_2)$ (see [5]).*

Using the notation in the previous section, Theorem 1.1 can be stated more precisely like this:

Theorem 3.3. *There exists $\delta_0 \in (0, m(J))$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0, \epsilon_0)$ the functional J_ϵ has at least $\text{cat}(M)$ critical points $u \in H_2^1(M)$ satisfying $J_\epsilon(u) < m(J) + \delta$ and at least one critical point with $J_\epsilon(u) > m(J) + \delta$.*

This theorem is a consequence of the following classical result (see for example [6]):

Theorem 3.4. *Let J be a C^1 real functional on a complete $C^{1,1}$ submanifold N of a Banach space. If J is bounded below and satisfies the (PS) condition then it has at least $\text{cat}(J^d)$ critical points in J^d , where $J^d := \{u \in N : J(u) < d\}$, and at least one critical point $u \notin J^d$.*

More precisely, Theorem 3.3 follows from the previous theorem, Remark 3.2 and the following proposition:

Proposition 3.5. *There exists $\delta_0 \in (0, m(J))$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0, \epsilon_0)$ we have*

$$\text{cat}(M) \leq \text{cat}(\Sigma_{\epsilon, \delta}).$$

In order to prove this we will present two suitable functions g_1 and g_2 .

By the embedding theorem, we assume that M is embedded in \mathbb{R}^N , with $N \geq 2n$.

Definition 3.6. *We define the radius of topological invariance $r(M)$ of M as*

$$r(M) := \sup\{\rho > 0 \mid \text{cat}(M_\rho) = \text{cat}(M)\},$$

where $M_\rho := \{z \in \mathbb{R}^N \mid d(z, M) < \rho\}$.

We can now show a function $\phi_\epsilon : M \rightarrow \Sigma_{\epsilon, \delta}$ and a function $\beta : \Sigma_{\epsilon, \delta} \rightarrow M_r$, with $0 < r < r(M)$ such that

$$I_\epsilon := \beta \circ \phi_\epsilon : M \rightarrow M_r \tag{3.1}$$

is well defined and homotopic to the identity on M .

4 The function ϕ_ϵ

Next lemma presents some properties of the Nehari manifold.

Lemma 4.1. *(i) The set \mathcal{N}_ϵ (resp. \mathcal{N}) is a C^1 manifold.*

(ii) For all not constant $u \in H_1^2(M)$ (for all $v \in \mathcal{D}^{1,2}(\mathbb{R}^n)$, $v \neq 0$), there exists a unique $t_\epsilon(u) > 0$ ($t(v) > 0$) such that $t_\epsilon(u)u \in \mathcal{N}_\epsilon$ ($t(v)v \in \mathcal{N}$) and $J_\epsilon(t_\epsilon(u)u)$ ($J(t(v)v)$) is the maximum value of $J_\epsilon(tu)$ ($J(tv)$) for $t \geq 0$.

(iii) The dependence of $t_\epsilon(u)$ on u (of $t(v)$ on v) is C^1 .

For the proof see Lemma 3.1 in [8].

Let U be the function defined in Section 2. We write

$$\tilde{U}_{\frac{R}{\epsilon}} = U(z) \quad \text{with} \quad z \in \mathbb{R}^n \quad \text{such that} \quad |z| = \frac{R}{\epsilon}.$$

For any $x_0 \in M$ and $\epsilon > 0$, we consider the function on M

$$W_{x_0, \epsilon}(x) := \begin{cases} U_\epsilon(\exp_{x_0}^{-1}(x)) - \tilde{U}_{\frac{R}{\epsilon}} & \text{if } x \in B_g(x_0, R), \\ 0 & \text{otherwise,} \end{cases} \tag{4.1}$$

where R is chosen as in Remark 2.1 (ii).

The function $W_{x_0, \epsilon}$ is in $H_1^2(M)$ and is not identically zero. Then, by the previous lemma, we can define

$$\begin{aligned} \phi_\epsilon &: M \longrightarrow \mathcal{N}_\epsilon \\ x_0 &\longmapsto t_\epsilon(W_{x_0, \epsilon}(x))W_{x_0, \epsilon}(x). \end{aligned} \quad (4.2)$$

The choice of the function ϕ_ϵ different from the one in [3] has been made for the function U can be not in $L^2(\mathbb{R}^n)$.

Proposition 4.2. *For any $\epsilon > 0$ the map $\phi_\epsilon : M \rightarrow \mathcal{N}_\epsilon$ is continuous. For any $\delta > 0$ there exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$*

$$\phi_\epsilon(x_0) \in \Sigma_{\epsilon, \delta}$$

for all $x_0 \in M$.

Proof. (I) The map $\phi_\epsilon : M \rightarrow \mathcal{N}_\epsilon$ is continuous.

By Lemma 4.1 (iii), it is enough to prove that

$$\lim_{k \rightarrow \infty} \|W_{x_k, \epsilon} - W_{\hat{x}, \epsilon}\|_{H_2^1(M)} = 0.$$

for any sequence $\{x_k\}$ in M , converging to \hat{x} .

We choose a finite atlas \mathcal{C} for M , which contains the chart $C = B_g(\hat{x}, R)$. The functions $W_{x_k, \epsilon}$ and $W_{\hat{x}, \epsilon}$ have support respectively on $B_g(x_k, R)$ and on $B_g(\hat{x}, R)$. Since $x_k \rightarrow \hat{x}$ the set $Z_k = [B_g(x_k, R) \setminus B_g(\hat{x}, R)] \cup [B_g(\hat{x}, R) \setminus B_g(x_k, R)]$ is such that $\mu_g(Z_k) \rightarrow 0$ as $k \rightarrow \infty$. Then we have

$$\int_{Z_k} |\nabla (W_{x_k, \epsilon}(x) - W_{\hat{x}, \epsilon}(x))|_g^2 d\mu_g \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We still have to check the integral on $B_g(x_k, R) \cap B_g(\hat{x}, R)$. We write $A_k = \exp_{\hat{x}}^{-1}(B_g(x_k, R) \cap B_g(\hat{x}, R))$ and $\eta_k(z) = \exp_{x_k}^{-1}(\exp_{\hat{x}}(z))$

$$\begin{aligned} \int_{\exp_{\hat{x}}^{-1}(A_k)} |\nabla [W_{x_k, \epsilon}(x) - W_{\hat{x}, \epsilon}(x)]|_g^2 d\mu_g &= \int_{A_k} |\nabla [U_\epsilon(\eta_k(z)) - U_\epsilon(z)]|_{g_{\hat{x}}(z)}^2 |g_{\hat{x}}(z)|^{\frac{1}{2}} dz \\ &\leq \frac{H^{\frac{n}{2}}}{h} \int_{A_k} |\nabla [U_\epsilon(\eta_k(z)) - U_\epsilon(z)]|^2 dz. \end{aligned}$$

Since $\eta_k(z)$ tends point-wise to z and ∇U_ϵ is continuous, $|\nabla [U_\epsilon(\eta_k(z)) - U_\epsilon(z)]|^2$ tends pointwise to zero. Applying Lebesgue theorem, we obtain that

$$\int_M |\nabla [W_{x_k, \epsilon}(x) - W_{\hat{x}, \epsilon}(x)]|_g^2 d\mu_g \rightarrow 0.$$

In an analogous way we have that $\|W_{x_k, \epsilon} - W_{\hat{x}, \epsilon}\|_{L^2(M)}^2$ tends to zero.

(II) *The limit of $\frac{\epsilon^2}{\epsilon^n} \int_M |\nabla W_{x_0, \epsilon}(x)|_g^2 d\mu_g$ is $\|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2$.*

To prove the second statement of this proposition, first we show that

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon^2}{\epsilon^n} \int_M |\nabla W_{x_0, \epsilon}(x)|_g^2 d\mu_g = \|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2 \quad (4.3)$$

uniformly with respect to $x_0 \in M$.

We evaluate the following:

$$\begin{aligned} & \left| \frac{\epsilon^2}{\epsilon^n} \int_M |\nabla W_{x_0, \epsilon}|_g^2 d\mu_g - \int_{\mathbb{R}^n} |\nabla U|^2 dz \right| \\ &= \left| \frac{\epsilon^2}{\epsilon^n} \int_{B_g(x_0, R)} |\nabla [U_\epsilon(\exp_{x_0}^{-1}(x))]|_g^2 d\mu_g - \int_{\mathbb{R}^n} |\nabla U|^2 dz \right| \\ &= \left| \frac{\epsilon^2}{\epsilon^n} \int_{B(0, R)} |\nabla U_\epsilon(z)|_{g_{x_0}(z)}^2 |g_{x_0}(z)|^{\frac{1}{2}} dz - \int_{\mathbb{R}^n} |\nabla U|^2 dz \right|. \end{aligned}$$

Changing variables, we obtain

$$\left| \int_{\mathbb{R}^n} \left(\chi_{B(0, \frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \delta^{ij} \right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} dz \right|,$$

where $\chi_{B(0, \frac{R}{\epsilon})}(z)$ denotes the characteristic function of the set $B(0, \frac{R}{\epsilon})$ and where δ^{ij} is the Kronecker delta (it takes value 0 for $i \neq j$ and 1 for $i = j$). The previous integral is bounded from above by the following sum

$$\begin{aligned} & \left| \int_{B(0, T)} \left(\chi_{B(0, \frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \delta^{ij} \right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} dz \right| \\ &+ \left| \int_{\mathbb{R}^n \setminus B(0, T)} \left(\chi_{B(0, \frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \delta^{ij} \right) \frac{\partial U}{\partial z_i} \frac{\partial U}{\partial z_j} dz \right|, \end{aligned}$$

with $T > 0$. It is easy to see that the second addendum vanishes as $T \rightarrow \infty$. As regards the first addendum, fixed T , by compactness of the manifold M and regularity of the Riemannian metric g the limit

$$\lim_{\epsilon \rightarrow 0} \left| \chi_{B(0, \frac{R}{\epsilon})}(z) g_{x_0}^{ij}(\epsilon z) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - \delta^{ij} \right| = 0$$

holds true uniformly with respect to $x_0 \in M$ and $z \in B(0, T)$ and (4.3) is proved.

(III) *There exists $t_1 > 0$ such that $t_\epsilon(W_{x_0, \epsilon}) \geq t_1$ for any $\epsilon \in (0, 1]$ and $x_0 \in M$.*

Let $g_{\epsilon, u}(t) = J_\epsilon(tu)$. By Lemma 4.1 (ii), it is enough to find $t_1 > 0$ such that for all $t \in [0, t_1]$ $g'_{\epsilon, W_{x_0, \epsilon}}(t) > 0$ for all $\epsilon \leq 1$ and for all $x_0 \in M$. Then we

look for a lower bound of $g'_{\epsilon, W_{x_0, \epsilon}}(t)$:

$$\begin{aligned}
g'_{\epsilon, W_{x_0, \epsilon}}(t) &= \frac{\epsilon^2 t}{\epsilon^n} \int_M |\nabla W_{x_0, \epsilon}|_g^2 d\mu_g - \frac{1}{\epsilon^n} \int_M f'(tW_{x_0, \epsilon}) W_{x_0, \epsilon} d\mu_g \\
&= \frac{1}{\epsilon^n} \int_{B(0, R)} [\epsilon^2 t |\nabla U_\epsilon(z)|_{g_{x_0}(z)}^2 - f'(tU_\epsilon(z) - t\tilde{U}_{\frac{R}{\epsilon}})(U_\epsilon(z) - \tilde{U}_{\frac{R}{\epsilon}})] |g_{x_0}(z)|^{\frac{1}{2}} dz \\
&= \int_{B(0, \frac{R}{\epsilon})} [t |\nabla U(z)|_{g_{x_0}(\epsilon z)}^2 - f'(tU(z) - t\tilde{U}_{\frac{R}{\epsilon}})(U(z) - \tilde{U}_{\frac{R}{\epsilon}})] |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz.
\end{aligned}$$

Using Remark 2.2, the fact that $\epsilon \leq 1$ and the properties of f (f1) and (f2), we obtain the following inequality:

$$\begin{aligned}
g'_{\epsilon, W_{x_0, \epsilon}}(t) &> \frac{h^{\frac{n}{2}} t}{H} \int_{B(0, R)} |\nabla U(z)|^2 dz - c_1 H^{\frac{n}{2}} \int_{G_{t, \epsilon}} t^{p-1} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz \\
&\quad - c_1 H^{\frac{n}{2}} \int_{L_{t, \epsilon}} t^{q-1} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz,
\end{aligned}$$

where $G_{t, \epsilon} = \left\{ z \in B\left(0, \frac{R}{\epsilon}\right) \mid t(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) \geq 1 \right\}$ and $L_{t, \epsilon} = \left\{ z \in B\left(0, \frac{R}{\epsilon}\right) \mid t(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) \leq 1 \right\}$. If $t \leq 1$, the following inclusions hold:

$$\begin{aligned}
G_{t, \epsilon} &\subset \left\{ z \in B\left(0, \frac{R}{\epsilon}\right) \mid U(z) - \tilde{U}_{\frac{R}{\epsilon}} \geq 1 \right\} \\
&\subset \left\{ z \in B\left(0, \frac{R}{\epsilon}\right) \mid U(z) \geq 1 \right\} \subset \{z \in \mathbb{R}^n \mid U(z) \geq 1\} = G.
\end{aligned}$$

By these inclusions and the fact that $|U(z) - \tilde{U}_{\frac{R}{\epsilon}}| \leq |U(z)|$,

$$\int_{G_{t, \epsilon}} t^{p-1} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz \leq \int_G t^{p-1} |U(z)|^p dz.$$

Let $L = \{z \in \mathbb{R}^n \mid U(z) \leq 1\}$. We have

$$\begin{aligned}
\int_{L_{t, \epsilon}} t^{q-1} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz &= \int_{L \cap B(0, \frac{R}{\epsilon})} t^{q-1} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz + \int_{L_{t, \epsilon} \setminus L} t^{q-1} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz \\
&\leq \int_L t^{q-1} |U(z)|^q dz + \int_{L_{t, \epsilon} \setminus L} t^{p-1} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz \\
&\leq \int_L t^{q-1} |U(z)|^q dz + \int_G t^{p-1} |U(z)|^p dz.
\end{aligned}$$

We conclude that

$$g'_{\epsilon, W_{x_0, \epsilon}}(t) > \gamma_1 t - \gamma_2 t^{p-1} - \gamma_3 t^{q-1}$$

with γ_1, γ_3 positive constants and γ_2 nonnegative constant. This proves the existence of t_1 .

(IV) There exists $t_2 > 0$ such that $t_\epsilon(W_{x_0,\epsilon}) \leq t_2$ for any $\epsilon \in (0, 1]$ and $x_0 \in M$.

If u is a function in the Nehari manifold \mathcal{N}_ϵ , we have that $J_\epsilon(u) = \tilde{F}_{\epsilon,M}(u)$, as defined in (2.9). Then by property (f1) $J_\epsilon(u)$ is positive. By Lemma 4.1 (ii), it is enough to find $t_2 > 0$ such that for all $t \geq t_2$ $J_\epsilon(tW_{x_0,\epsilon}) < 0$ for all $\epsilon \leq 1$ and for all $x_0 \in M$. Then we look for an upper bound of $J_\epsilon(tW_{x_0,\epsilon})$:

$$\begin{aligned}
J_\epsilon(tW_{x_0,\epsilon}) &= \frac{\epsilon^2 t^2}{2\epsilon^n} \int_M |\nabla W_{x_0,\epsilon}|_g^2 d\mu_g - \frac{1}{\epsilon^n} \int_M f(tW_{x_0,\epsilon}) d\mu_g \\
&= \frac{1}{\epsilon^n} \int_{B(0,R)} \left[\frac{\epsilon^2 t^2}{2} |\nabla U_\epsilon(z)|_{g_{x_0}(z)}^2 - f\left(tU_\epsilon(z) - t\tilde{U}_{\frac{R}{\epsilon}}\right) \right] |g_{x_0}(z)|^{\frac{1}{2}} dz \\
&= \int_{B(0,\frac{R}{\epsilon})} \left[\frac{t^2}{2} |\nabla U(z)|_{g_{x_0}(\epsilon z)}^2 - f\left(tU(z) - t\tilde{U}_{\frac{R}{\epsilon}}\right) \right] |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz \\
&\leq \frac{H^{\frac{n}{2}} t^2}{2h} \|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2 - c_0 h^{\frac{n}{2}} \int_{G_{t,\epsilon}} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz - c_0 h^{\frac{n}{2}} \int_{L_{t,\epsilon}} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz.
\end{aligned}$$

If we consider $t \geq 1$ and $\tilde{U}_R = U(z)$ with $z \in \mathbb{R}^n$ such that $|z| = R$, there holds

$$\begin{aligned}
&\int_{G_{t,\epsilon}} t^p |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz + \int_{L_{t,\epsilon}} t^q |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz \\
&\geq t^p \left[\int_{G_{1,\epsilon}} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz + \int_{G_{t,\epsilon} \setminus G_{1,\epsilon}} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz \right. \\
&\quad \left. + \int_{L_{1,\epsilon}} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz - \int_{L_{1,\epsilon} \setminus L_{t,\epsilon}} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz \right] \\
&\geq t^p \left[\int_{G_{1,\epsilon}} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz + \int_{L_{1,\epsilon}} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz \right] \\
&\geq t^p \left[\int_{G_{1,\epsilon} \cap B(0,R)} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^p dz + \int_{L_{1,\epsilon} \cap B(0,R)} |U(z) - \tilde{U}_{\frac{R}{\epsilon}}|^q dz \right] \\
&\geq t^p \left[\int_{G_{1,\epsilon} \cap B(0,R)} |U(z) - \tilde{U}_R|^p dz + \int_{L_{1,\epsilon} \cap B(0,R)} |U(z) - \tilde{U}_R|^q dz \right] \\
&= t^p \left[\int_{G_{1,1}} |U(z) - \tilde{U}_R|^p dz + \int_{G_{1,\epsilon} \cap B(0,R) \setminus G_{1,1}} |U(z) - \tilde{U}_R|^p dz \right. \\
&\quad \left. + \int_{L_{1,1}} |U(z) - \tilde{U}_R|^q dz - \int_{L_{1,1} \setminus L_{1,\epsilon}} |U(z) - \tilde{U}_R|^q dz \right] \\
&\geq t^p \left[\int_{G_{1,1}} |U(z) - \tilde{U}_R|^p dz + \int_{L_{1,1}} |U(z) - \tilde{U}_R|^q dz \right].
\end{aligned}$$

So $J_\epsilon(tW_{x_0,\epsilon}) \leq \gamma_4 t^2 - \gamma_5 t^p$ with γ_4, γ_5 positive constants and for t big enough it is negative.

(V) The parameter $t_\epsilon(W_{x_0,\epsilon})$ tends to 1 for ϵ tending to zero uniformly with respect to $x_0 \in M$.

By the previous steps $t_\epsilon(W_{x_0,\epsilon}) \in [t_1, t_2]$ for any $\epsilon \in (0, 1]$ and $x_0 \in M$. Let us write $t_{x_0,\epsilon} = t_\epsilon(W_{x_0,\epsilon})$. Then there exists a sequence $\epsilon_k \rightarrow 0$ for $k \rightarrow \infty$ such that t_{x_0,ϵ_k} converges to $t_{x_0}^*$. Then by step (II) we have $\lim_{k \rightarrow \infty} \frac{\epsilon_k^2}{\epsilon_k^n} \int_M |t_{x_0,\epsilon_k} \nabla W_{x_0,\epsilon_k}(x)|_g^2 d\mu_g = \|t_{x_0}^* U\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2$. By definition we have

$$\begin{aligned} & \frac{1}{\epsilon_k^n} \int_M f'(t_{x_0,\epsilon_k} W_{x_0,\epsilon_k}) t_{x_0,\epsilon_k} W_{x_0,\epsilon_k} d\mu_g \\ &= \frac{1}{\epsilon_k^n} \int_{B(0,R)} f'(t_{x_0,\epsilon_k} (U_{\epsilon_k}(z) - \tilde{U}_{\frac{R}{\epsilon_k}})) t_{x_0,\epsilon_k} (U_{\epsilon_k}(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) |g_{x_0}(z)|^{\frac{1}{2}} dz \\ &= \int_{B(0,\frac{R}{\epsilon_k})} f'(t_{x_0,\epsilon_k} (U(z) - \tilde{U}_{\frac{R}{\epsilon_k}})) t_{x_0,\epsilon_k} (U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) |g_{x_0}(\epsilon_k z)|^{\frac{1}{2}} dz \\ &= \int_{\mathbb{R}^n} \chi_{B(0,\frac{R}{\epsilon_k})}(z) f'(t_{x_0,\epsilon_k} (U(z) - \tilde{U}_{\frac{R}{\epsilon_k}})) t_{x_0,\epsilon_k} (U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) |g_{x_0}(\epsilon_k z)|^{\frac{1}{2}} dz. \end{aligned}$$

The integrand point-wise tends to $f'(t_{x_0}^* U(z)) t_{x_0}^* U(z)$ for k tending to infinity and is bounded from above by a function in $L^1(\mathbb{R}^n)$ as follows:

$$\begin{aligned} & \chi_{B(0,\frac{R}{\epsilon_k})}(z) f'(t_{x_0,\epsilon_k} (U(z) - \tilde{U}_{\frac{R}{\epsilon_k}})) t_{x_0,\epsilon_k} (U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) |g_{x_0}(\epsilon_k z)|^{\frac{1}{2}} \\ & \leq H^{\frac{n}{2}} \chi_{B(0,\frac{R}{\epsilon_k})}(z) f'(t_2(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}})) t_2(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) \\ & \leq \begin{cases} c_1 H^{\frac{n}{2}} t_2^p(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}})^p & \text{if } t_2(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) \geq 1 \text{ and } |z| \leq \frac{R}{\epsilon_k} \\ c_1 H^{\frac{n}{2}} t_2^q(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}})^q & \text{if } t_2(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) \leq 1 \text{ and } |z| \leq \frac{R}{\epsilon_k} \\ 0 & \text{otherwise} \end{cases} \\ & \leq \begin{cases} c_1 H^{\frac{n}{2}} t_2^p(U(z))^p & \text{if } t_2(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) \geq 1, U(z) \geq 1 \text{ and } |z| \leq \frac{R}{\epsilon_k} \\ c_1 H^{\frac{n}{2}} t_2^q(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}})^q & \text{if } t_2(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) \geq 1, U(z) < 1 \text{ and } |z| \leq \frac{R}{\epsilon_k} \\ c_1 H^{\frac{n}{2}} t_2^p(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}})^p & \text{if } t_2(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) < 1, U(z) \geq 1 \text{ and } |z| \leq \frac{R}{\epsilon_k} \\ c_1 H^{\frac{n}{2}} t_2^q(U(z))^q & \text{if } t_2(U(z) - \tilde{U}_{\frac{R}{\epsilon_k}}) < 1, U(z) < 1 \text{ and } |z| \leq \frac{R}{\epsilon_k} \\ 0 & \text{otherwise} \end{cases} \\ & \leq \begin{cases} c_1 H^{\frac{n}{2}} t_2^p(U(z))^p & \text{if } U(z) \geq 1 \\ c_1 H^{\frac{n}{2}} t_2^q(U(z))^q & \text{if } U(z) < 1 \end{cases} \\ & \leq \frac{c_1 H^{\frac{n}{2}} t_2^q}{c_0} f(U(z)). \end{aligned}$$

Then by Lebesgue theorem $\lim_{k \rightarrow \infty} \frac{1}{\epsilon_k^n} \int_M f'(t_{x_0,\epsilon_k} W_{x_0,\epsilon_k}) t_{x_0,\epsilon_k} W_{x_0,\epsilon_k} d\mu_g = \int_{\mathbb{R}^n} f'(t_{x_0}^* U(z)) t_{x_0}^* U(z) dz$. By the fact that $U \in \mathcal{N}$ and $\|t_{x_0}^* U\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} f'(t_{x_0}^* U(z)) t_{x_0}^* U(z) dz$, we conclude that $t_{x_0}^* = 1$.

To prove that the convergence is uniform with respect to $x_0 \in M$, we show that $\lim_{\epsilon \rightarrow 0} \sup_{x \in M} |t_{x,\epsilon} - 1| = 0$. For any ϵ there exists $x(\epsilon) \in M$ such that $\sup_{x \in M} |t_{x,\epsilon} - 1| = |t_{x(\epsilon),\epsilon} - 1|$. By compactness there exists a sequence $\epsilon_k \rightarrow 0$

for $k \rightarrow \infty$ such that $x(\epsilon_k)$ tends to $x_* \in M$. Let us fix $\eta > 0$. There exists k_0 such that for all $k \geq k_0$ $|t_{x_*, \epsilon_k} - 1| < \frac{\eta}{3}$. Possibly increasing k_0 we also have that for all $k \geq k_0$ and $h > k$ $|t_{x(\epsilon_k), \epsilon_k} - t_{x(\epsilon_h), \epsilon_k}| < \frac{\eta}{3}$. Finally there exists h_0 such that for all $h \geq h_0$ $|t_{x(\epsilon_h), \epsilon_k} - t_{x_*, \epsilon_k}| < \frac{\eta}{3}$. Summing the three terms one has that $|t_{x(\epsilon_k), \epsilon_k} - 1| < \eta$ for all $k \geq k_0$.

(VI) The limit of $\frac{1}{\epsilon^n} \int_M f(t_{x_0, \epsilon} W_{x_0, \epsilon}) d\mu_g$ is $\int_{\mathbb{R}^n} f(U) dz$.

Changing variables and using mean value theorem, we have

$$\begin{aligned} \frac{1}{\epsilon^n} \int_M f(t_{x_0, \epsilon} W_{x_0, \epsilon}) d\mu_g &= \int_{B(0, \frac{R}{\epsilon})} \left[f(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) \right. \\ &\quad \left. + (t_{x_0, \epsilon} - 1) f'(\Theta_{x_0, \epsilon}(z)(U(z) - \tilde{U}_{\frac{R}{\epsilon}}))(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) \right] |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz, \end{aligned}$$

where $\Theta_{x_0, \epsilon}(z) = (\theta_{x_0, \epsilon}(z)t_{x_0, \epsilon} + 1 - \theta_{x_0, \epsilon}(z))$ with a suitable $0 < \theta_{x_0, \epsilon}(z) < 1$. We want to prove that

$$\begin{aligned} \int_{B(0, \frac{R}{\epsilon})} f(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz &\xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{R}^n} f(U) dz \\ \int_{B(0, \frac{R}{\epsilon})} (t_{x_0, \epsilon} - 1) f'(\Theta_{x_0, \epsilon}(z)(U(z) - \tilde{U}_{\frac{R}{\epsilon}}))(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz &\xrightarrow{\epsilon \rightarrow 0} 0 \end{aligned} \quad (4.4)$$

uniformly with respect to $x_0 \in M$.

It is easy to see that

$$\int_{B(0, \frac{R}{\epsilon})} f(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) \left| |g_{x_0}(\epsilon z)|^{\frac{1}{2}} - 1 \right| dz \xrightarrow{\epsilon \rightarrow 0} 0$$

uniformly with respect to $x_0 \in M$. The function $\chi_{B(0, \frac{R}{\epsilon})}(z) f(U(z) - \tilde{U}_{\frac{R}{\epsilon}})$ tends pointwise to $f(U(z))$ for any $z \in \mathbb{R}^n$. Moreover

$$\begin{aligned} \chi_{B(0, \frac{R}{\epsilon})}(z) f(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) &\leq \begin{cases} \frac{c_1}{\mu} (U(z) - \tilde{U}_{\frac{R}{\epsilon}})^p & \text{if } U(z) - \tilde{U}_{\frac{R}{\epsilon}} \geq 1, |z| \leq \frac{R}{\epsilon} \\ \frac{c_1}{\mu} (U(z) - \tilde{U}_{\frac{R}{\epsilon}})^q & \text{if } U(z) - \tilde{U}_{\frac{R}{\epsilon}} \leq 1, |z| \leq \frac{R}{\epsilon} \\ 0 & \text{otherwise} \end{cases} \\ &\leq \begin{cases} \frac{c_1}{\mu} (U(z) - \tilde{U}_{\frac{R}{\epsilon}})^p & \text{if } U(z) \geq 1, |z| \leq \frac{R}{\epsilon} \\ \frac{c_1}{\mu} (U(z) - \tilde{U}_{\frac{R}{\epsilon}})^q & \text{if } U(z) < 1, |z| \leq \frac{R}{\epsilon} \\ 0 & \text{otherwise} \end{cases} \\ &\leq \begin{cases} \frac{c_1}{\mu} (U(z))^p & \text{if } U(z) \geq 1 \\ \frac{c_1}{\mu} (U(z))^q & \text{if } U(z) \leq 1 \end{cases} \\ &\leq \frac{c_1}{c_0 \mu} f(U(z)) \end{aligned}$$

and by Lebesgue theorem we obtain the first limit in (4.4). The function of t $f'(tu)u$ is increasing in t , since its derivative is $f''(tu)u^2 > 0$. Then we have

$$\begin{aligned} \int_{B(0, \frac{R}{\epsilon})} f'(\Theta_{x_0, \epsilon}(z)(U(z) - \tilde{U}_{\frac{R}{\epsilon}}))(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz \\ < H^{\frac{n}{2}} \int_{B(0, \frac{R}{\epsilon})} f'((t_2 + 1)(U(z) - \tilde{U}_{\frac{R}{\epsilon}}))(U(z) - \tilde{U}_{\frac{R}{\epsilon}}) dz. \end{aligned}$$

By the usual standard inequalities, the previous integral is bounded from above by $\frac{c_1 H^{\frac{n}{2}}}{c_0(t_2+1)} \int_{\mathbb{R}^n} f((t_2+1)U(z))dz$ and the second limit in (4.4) is proved, because of (V).

(VII) Conclusion.

By (II), (V) and (VI) we obtain that $J_\epsilon(\phi_\epsilon(x_0))$ tends to $J(U) = m(J)$ for ϵ tending to zero uniformly with respect to x_0 . This completes the proof. \square

Remark 4.3. *By the previous proposition, in particular we know that, given $\delta > 0$, for any positive ϵ sufficiently small $\Sigma_{\epsilon,\delta}$ is not empty.*

5 The function β

Given a function $u \in L^p(M)$, $u \not\equiv 0$, it is possible to define its centre of mass $\beta(u) \in \mathbb{R}^N$ by

$$\beta(u) = \frac{\int_M x \Phi(u) d\mu_g}{\int_M \Phi(u) d\mu_g}, \quad (5.1)$$

where

$$\Phi(u) = \frac{1}{2} f'(u)u - f(u). \quad (5.2)$$

By the properties of f , $\Phi(s) > 0$ for all $s \neq 0$. To prove that $\beta : \Sigma_{\epsilon,\delta} \rightarrow M_{r(M)}$ (see Section 3 and Definition 3.6), we use the fact that the functions in $\Sigma_{\epsilon,\delta}$ concentrate for ϵ and δ tending to zero.

First of all we find a positive inferior bound for the functional J_ϵ on the Nehari manifold. Let us denote

$$m_\epsilon = \inf_{u \in \mathcal{N}_\epsilon} J_\epsilon(u). \quad (5.3)$$

It is easy to see that

$$\inf_{u \in \mathcal{N}_\epsilon} \|u\|_{H_2^1(M)} > 0$$

(the proof is analogous to Lemma 3.2 of [8]) and, since the manifold M is compact, that the infimum m_ϵ is achieved.

Lemma 5.1. *There exist positive constants α and ϵ_0 such that for any $0 < \epsilon < \epsilon_0$ the inequality $m_\epsilon \geq \alpha$ holds.*

To prove this lemma we need the following technical lemma (for the proof see the Appendix).

Lemma 5.2. *For any $r \in (0, r(M))$, there exist constants $k_1, k_2, k_3, k_4 > 0$ such that for any $u \in H_2^1(M)$ there exists $v \in \mathcal{D}^{1,2}(M_r)$ such that $v|_M \equiv u$ and*

$$\|v\|_{\mathcal{D}^{1,2}(M_r)}^2 \leq k_1 \int_M |\nabla u|_g^2 d\mu_g, \quad (5.4)$$

$$\int_{M_r} f(v(z)) dz \geq k_2 \int_M f(u(x)) d\mu_g, \quad (5.5)$$

$$\int_{M_r} f(v(z)) dz \leq k_3 \int_M f(u(x)) d\mu_g, \quad (5.6)$$

$$\|v\|_{L^2(M_r)}^2 \geq k_4 \|u\|_{L^2(M)}^2. \quad (5.7)$$

Proof of Lemma 5.1. By definition m_ϵ is the infimum of $J_\epsilon(u)$ on the Nehari manifold \mathcal{N}_ϵ . If $u \in \mathcal{N}_\epsilon$ we have

$$J_\epsilon(u) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \frac{\epsilon^2}{\epsilon^n} \int_M |\nabla u|_g^2 d\mu_g.$$

Rescaling u , it is easy to see that m_ϵ is greater than or equal to the infimum of the functional $\left(\frac{1}{2} - \frac{1}{\mu}\right) \frac{\epsilon^2}{\epsilon^n} t_\epsilon^2 \int_M |\nabla w|_g^2 d\mu_g$ on the set of the functions $w \in H_2^1(M)$ such that $\frac{1}{\epsilon^n} \int_M f(w) d\mu_g = 1$ and where $t_\epsilon = t_\epsilon(w)$ is as in (ii), Lemma 4.1. First of all, we check that there exists a constant $\tilde{\alpha} > 0$ and for such functions w it holds

$$\frac{\epsilon^2}{\epsilon^n} \int_M |\nabla w|_g^2 d\mu_g \geq \tilde{\alpha}.$$

By Lemma 5.2, for any function w there exists a function $v \in \mathcal{D}^{1,2}(M_r)$ such that (5.4) and (5.5) hold. We consider $\tilde{v} \in \mathcal{D}^{1,2}(\mathbb{R}^N)$, defined as $\tilde{v}(y) = v(y)$ for all $y \in M_r$ and $\tilde{v}(y) = 0$ for all $y \in \mathbb{R}^N \setminus M_r$. We can now consider the following rescaling $V(y) = \tilde{v}(\epsilon^\sigma y)$ with $\sigma = \frac{2n-(n-2)p}{2N-(N-2)p}$. In case the denominator is equal to 0, we can choose a bigger N . We have:

$$\|V\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 = \frac{\epsilon^{2\sigma}}{\epsilon^{N\sigma}} \|v\|_{\mathcal{D}^{1,2}(M_r)}^2 \quad \text{and} \quad \int_{\mathbb{R}^N} f(V(y)) dy = \frac{1}{\epsilon^{N\sigma}} \int_{M_r} f(v(y)) dy.$$

By these equalities, (5.4) and (5.5), we have

$$\begin{aligned} \frac{\epsilon^2}{\epsilon^n} \int_M |\nabla w|_g^2 d\mu_g &= \frac{\frac{\epsilon^2}{\epsilon^n} \int_M |\nabla w|_g^2 d\mu_g}{\left(\frac{1}{\epsilon^n} \int_M f(w) d\mu_g\right)^{\frac{2}{p}}} \geq \frac{k_2^{\frac{2}{p}}}{k_1} \frac{\frac{\epsilon^2}{\epsilon^n} \|v\|_{\mathcal{D}^{1,2}(M_r)}^2}{\left(\frac{1}{\epsilon^n} \int_{M_r} f(v) dy\right)^{\frac{2}{p}}} \\ &= \frac{k_2^{\frac{2}{p}}}{k_1} \frac{\frac{\epsilon^{(N-2)\sigma}}{\epsilon^{n-2}} \|V\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2}{\left(\frac{\epsilon^{N\sigma}}{\epsilon^n} \int_{\mathbb{R}^N} f(V) dy\right)^{\frac{2}{p}}} = \frac{k_2^{\frac{2}{p}}}{k_1} \frac{\|V\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2}{\left(\int_{\mathbb{R}^N} f(V) dy\right)^{\frac{2}{p}}}. \end{aligned} \quad (5.8)$$

We show now that for ϵ sufficiently small we have $\int_{\mathbb{R}^N} f(V) dy < 1$. In fact, by (5.6) there holds

$$\int_{\mathbb{R}^N} f(V) dy = \frac{1}{\epsilon^{N\sigma}} \int_{M_r} f(v(y)) dy \leq \frac{k_3}{\epsilon^{N\sigma}} \int_M f(w) d\mu_g = \frac{k_3 \epsilon^n}{\epsilon^{N\sigma}}.$$

By definition of σ $\lim_{N \rightarrow \infty} N\sigma = \frac{2n-(n-2)p}{2-p} < 0$ and so there exists N sufficiently big such that $n - N\sigma > 0$.

Since $\int_{\mathbb{R}^N} f(tV(y)) dy$ is an increasing function of t for positive t , there exists $t_* > 1$ such that $\int_{M_r} f(t_*V(y)) dy = 1$. Let $V_*(y) = t_*V(y)$ for any $y \in \mathbb{R}^N$. With the usual computation we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} f(V(y)) dy &= \int_{\mathbb{R}^N} f\left(\frac{1}{t_*}V_*(y)\right) dy \\ &< \frac{c_1}{\mu} \left(\int_{\{y \in \mathbb{R}^N \mid |V_*(y)| \geq t_*\}} \frac{1}{t_*^p} |V_*(y)|^p dy + \int_{\{y \in \mathbb{R}^N \mid |V_*(y)| \leq t_*\}} \frac{1}{t_*^q} |V_*(y)|^q dy \right) \\ &\leq \frac{c_1}{\mu} \left(\int_{\{y \in \mathbb{R}^N \mid |V_*(y)| \geq 1\}} \frac{1}{t_*^p} |V_*(y)|^p dy + \int_{\{y \in \mathbb{R}^N \mid |V_*(y)| \leq 1\}} \frac{1}{t_*^q} |V_*(y)|^q dy \right) \\ &\leq \frac{c_1}{c_0 \mu t_*^p} \int_{\mathbb{R}^N} f(V_*(y)) dy = \frac{c_1}{c_0 \mu t_*^p}. \end{aligned}$$

Concluding we have that the last term in (5.8) is equal to

$$\frac{k_2^{\frac{2}{p}}}{k_1} \frac{\frac{1}{t_*^2} \|V_*\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2}{\left(\int_{\mathbb{R}^N} f\left(\frac{1}{t_*}V_*\right) dy \right)^{\frac{2}{p}}} \geq \frac{k_2^{\frac{2}{p}}}{k_1} \left(\frac{c_0 \mu}{c_1} \right)^{\frac{2}{p}} \|V_*\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2$$

which is bounded from below because (see [10])

$$\inf_{\substack{V \in \mathcal{D}^{1,2}(\mathbb{R}^N) \\ \int_{\mathbb{R}^N} f(V) dy = 1}} \|V\|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}^2 = \hat{\alpha} > 0.$$

We still have to show that t_ϵ is bounded from below by a positive constant. By the properties (f1) and (f2) we have

$$\begin{aligned} \frac{1}{\epsilon^n} \int_M f'(t_\epsilon w) t_\epsilon w d\mu_g &< \frac{c_1}{\epsilon^n} \left[\int_{\{x \in M \mid |t_\epsilon w(x)| \geq 1\}} |t_\epsilon w(x)|^p d\mu_g + \int_{\{x \in M \mid |t_\epsilon w(x)| \leq 1\}} |t_\epsilon w(x)|^q d\mu_g \right] \\ &\leq \frac{c_1}{\epsilon^n} \left[\int_{\{x \in M \mid |w(x)| \geq 1\}} |t_\epsilon w(x)|^p d\mu_g + \int_{\{x \in M \mid |w(x)| \leq 1\}} |t_\epsilon w(x)|^q d\mu_g \right] \\ &\leq \frac{c_1 t_\epsilon^p}{c_0 \epsilon^n} \int_M f(w(x)) d\mu_g = \frac{c_1 t_\epsilon^p}{c_0}, \end{aligned}$$

where the last equality is due the property of the functions w . Since $t_\epsilon w \in \mathcal{N}_\epsilon$, $\frac{1}{\epsilon^n} \int_M f'(t_\epsilon w) t_\epsilon w d\mu_g = \frac{\epsilon^2 t_\epsilon^2}{\epsilon^n} \int_M |\nabla w|_g^2 d\mu_g$ and by the previous inequalities we have

$$t_\epsilon^{p-2} \geq \frac{c_0}{c_1} \frac{\epsilon^2}{\epsilon^n} \int_M |\nabla w|_g^2 d\mu_g \geq \frac{c_0}{c_1} \tilde{\alpha}$$

and this completes the proof. \square

In the following lemma for every function $u \in \mathcal{N}_\epsilon$ it is stated the existence of a point in the manifold where u in some sense concentrates.

Lemma 5.3. *Let \mathcal{C} be an atlas for M with open cover given by $B_g(x_i, R)$, $i = 1, \dots, A$, and partition of unity $\{\psi_i\}_{i=1\dots A}$. There exists a constant $\gamma > 0$ such that for any $0 < \epsilon < \epsilon_0$, where ϵ_0 is defined in Lemma 5.1, if $u \in \mathcal{N}_\epsilon$ there exists $i = i(u)$ such that*

$$\begin{aligned} \frac{1}{\epsilon^n} \int_{B_g(x_i, \frac{R}{2})} \left[\frac{1}{2} f'(u)u - f(u) \right] d\mu_g &\geq \gamma, \\ \frac{\epsilon^2}{2\epsilon^n} \int_{B_g(x_i, \frac{R}{2})} |\nabla u|_g^2 d\mu_g - \frac{1}{\epsilon^n} \int_{B_g(x_i, \frac{R}{2})} f(u) d\mu_g &\geq \gamma. \end{aligned} \quad (5.9)$$

Proof. Let u be in \mathcal{N}_ϵ . We assume that $\tilde{\mathcal{C}} = \{B_g(x_i, \frac{R}{2})\}_{i=1,\dots,A}$ is still an open cover (otherwise we complete \mathcal{C}). Let $\{\tilde{\psi}_i\}_{i=1\dots A}$ be a partition of unity subordinate to the atlas $\tilde{\mathcal{C}}$. If $\tilde{F}_{\epsilon, M}(u)$ is as in (2.9), it is possible to write

$$\begin{aligned} J_\epsilon(u) &= \left(\tilde{F}_{\epsilon, M}(u) \right)^{\frac{1}{2}} (J_\epsilon(u))^{\frac{1}{2}} \\ &= \left(\frac{1}{\epsilon^n} \sum_{i=1}^A \int_{B_g(x_i, \frac{R}{2})} \tilde{\psi}_i(x) \left[\frac{1}{2} f'(u(x))u(x) - f(u(x)) \right] d\mu_g \right)^{\frac{1}{2}} (J_\epsilon(u))^{\frac{1}{2}} \\ &\leq \sqrt{A} \max_{1 \leq i \leq A} \left(\tilde{F}_{\epsilon, B_g(x_i, \frac{R}{2})}(u) \right)^{\frac{1}{2}} (J_\epsilon(u))^{\frac{1}{2}} \end{aligned}$$

By this inequality and Lemma 5.1 we conclude that

$$\max_{1 \leq i \leq A} \tilde{F}_{\epsilon, B_g(x_i, \frac{R}{2})}(u) \geq \frac{1}{A} J_\epsilon(u) \geq \frac{\alpha}{A}.$$

The second equation in (5.9) is proved analogously. \square

In the following proposition the concentration property is better specified.

Proposition 5.4. *For any $\eta \in (0, 1)$ there exists $\delta_0 < m(J)$ such that, for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0, \epsilon_0)$ with every function $u \in \Sigma_{\epsilon, \delta}$ it is associated a point $x_0 = x_0(u)$ in M with the property*

$$\tilde{F}_{\epsilon, B_g(x_0, \frac{r(M)}{2})}(u) > \eta m(J).$$

The proof of this proposition needs the following lemmas. The first lemma we need is the splitting lemma proved in [8] (Lemma 4.1):

Lemma 5.5. *Let $\{v_k\}_{k \in \mathbb{N}} \subset \mathcal{N}$ be a sequence such that:*

$$\begin{aligned} J(v_k) &\rightarrow m(J) && \text{as } k \rightarrow \infty, \\ J'(v_k) &\rightarrow 0 \text{ in } \mathcal{D}^{1,2}(\mathbb{R}^n) && \text{as } k \rightarrow \infty. \end{aligned}$$

Then

- either v_k converges strongly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ to a ground state solution of (2.12)
- or there exist a sequence of points $\{y_k\}_{k \in \mathbb{N}} \subset \mathbb{R}^n$ with $|y_k| \rightarrow \infty$ as $k \rightarrow \infty$, a ground state solution U of (2.12) and a sequence of functions $\{v_k^0\}_{k \in \mathbb{N}}$ such that, up to a subsequence:
 - (i) $v_k(z) = v_k^0(z) + U(z - y_k)$ for all $z \in \mathbb{R}^n$;
 - (ii) $v_k^0 \rightarrow 0$ as $k \rightarrow \infty$ in $\mathcal{D}^{1,2}(\mathbb{R}^n)$.

Lemma 5.6. *Let ϵ_k and δ_k be two positive sequences tending to zero for k tending to infinity. For any $k \in \mathbb{N}$ let u_k be a function in $\Sigma_{\epsilon_k, \delta_k}$ such that for any $u \in H_2^1(M)$*

$$|J'_{\epsilon_k}(u_k)(u)| = o\left(\frac{\epsilon_k}{\epsilon_k^{\frac{n}{2}}}\|u\|_{H_2^1(M)}\right).$$

There exist a sequence $\{x_k\}_{k \in \mathbb{N}}$ of points in M and a sequence of functions w_k on \mathbb{R}^n , defined as

$$w_k(z) = u_k(\exp_{x_k}(\epsilon_k z)) \chi_{\frac{R}{\epsilon_k}}(|z|), \quad (5.10)$$

such that the following properties hold:

- (i) *There exists $w \in \mathcal{D}^{1,2}(\mathbb{R}^n)$ such that, up to a subsequence, w_k tends to w weakly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ and strongly in $L_{loc}^p(\mathbb{R}^n)$.*
- (ii) *The function w is a weak solution of $-\Delta w = f'(w)$ on \mathbb{R}^n .*
- (iii) *The function w is a ground state solution.*
- (iv) *The following equality holds*

$$\lim_{k \rightarrow \infty} J_{\epsilon_k}(u_k) = m(J).$$

Proof. To get started we consider x_k to be the points in M such that u_k has the property (5.9). We will be more precise in point (iii).

(i) It is sufficient to prove that the sequence w_k is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$. We write:

$$\begin{aligned} \|w_k\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^2 &= \int_{B(0, \frac{R}{\epsilon_k})} |\nabla w_k(z)|^2 dz \\ &\leq 2 \int_{B(0, \frac{R}{\epsilon_k})} |\nabla[u_k(\exp_{x_k}(\epsilon_k z))]|^2 \left[\chi_{\frac{R}{\epsilon_k}}(|z|)\right]^2 dz \\ &\quad + 2 \int_{B(0, \frac{R}{\epsilon_k})} \left[\chi'_{\frac{R}{\epsilon_k}}(|z|)\right]^2 [u_k(\exp_{x_k}(\epsilon_k z))]^2 dz = I_1 + I_2. \end{aligned}$$

We consider the following inequality:

$$\begin{aligned}
\frac{\epsilon_k^2}{\epsilon_k^n} \int_M |\nabla u_k|_g^2 d\mu_g &\geq \frac{\epsilon_k^2}{\epsilon_k^n} \int_{B_g(x_k, R)} |\nabla u_k|_g^2 d\mu_g \\
&= \frac{\epsilon_k^2}{\epsilon_k^n} \int_{B(0, R)} |\nabla u_k(\exp_{x_k}(z))|_{g_{x_k}(z)}^2 |g_{x_k}(z)|^{\frac{1}{2}} dz \\
&= \int_{B(0, \frac{R}{\epsilon_k})} |\nabla u_k(\exp_{x_k}(\epsilon_k z))|_{g_{x_k}(\epsilon_k z)}^2 |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}} dz \\
&\geq \frac{h^{\frac{n}{2}}}{H} \int_{B(0, \frac{R}{\epsilon_k})} |\nabla u_k(\exp_{x_k}(\epsilon_k z))|^2 dz \geq \frac{h^{\frac{n}{2}}}{2H} I_1.
\end{aligned} \tag{5.11}$$

Moreover the following inequality holds

$$\begin{aligned}
I_2 &\leq \frac{2\chi_0^2 \epsilon_k^2}{R^2} \int_{B(0, \frac{R}{\epsilon_k})} [u_k(\exp_{x_k}(\epsilon_k z))]^2 dz \\
&= \frac{2\chi_0^2 \epsilon_k^2}{R^2 \epsilon_k^n} \int_{B(0, R)} [u_k(\exp_{x_k}(z))]^2 dz \\
&\leq \frac{2\chi_0^2 \epsilon_k^2}{h^{\frac{n}{2}} R^2 \epsilon_k^n} \int_{B_g(x_k, R)} (u_k(x))^2 d\mu_g.
\end{aligned} \tag{5.12}$$

By (5.11) and (5.12), we have that the sum $I_1 + I_2$ is bounded by a constant times $\frac{\epsilon_k^2}{\epsilon_k^n} \|u_k\|_{H_2^1(M)}^2$. We show then that this quantity must be bounded. Since $u_k \in \Sigma_{\epsilon_k, \delta_k}$ and

$$J_{\epsilon_k}(u_k) \geq \left(\frac{1}{2} - \frac{1}{\mu}\right) \frac{\epsilon_k^2}{\epsilon_k^n} \int_M |\nabla u_k|_g^2 d\mu_g,$$

the right hand side of the preceding inequality must be bounded. We still have to check that $\frac{\epsilon_k^2}{\epsilon_k^n} \|u_k\|_{L^2(M)}^2$ is bounded too. In fact, by (5.7) in Lemma 5.2 we have a sequence v_k of functions in $\mathcal{D}^{1,2}(M_r)$ and

$$\frac{\epsilon_k^2}{\epsilon_k^n} \|u_k\|_{L^2(M)}^2 \leq \frac{\epsilon_k^2}{k_4 \epsilon_k^n} \|v_k\|_{L^2(M_r)}^2 \leq \frac{C \epsilon_k^2}{k_4 \epsilon_k^n} \|v_k\|_{\mathcal{D}^{1,2}(M_r)}^2 \leq \frac{C k_1 \epsilon_k^2}{k_4 \epsilon_k^n} \int_M |\nabla u_k|_g^2 d\mu_g,$$

where C is the constant in the Poincaré inequality and we have used (5.4) in the last inequality.

(ii) First of all we prove that for any $\xi \in C_0^\infty(\mathbb{R}^n)$ $J'(w_k)(\xi)$ tends to zero for k tending to infinity:

$$\begin{aligned}
J'(w_k)(\xi) &= \int_{\mathbb{R}^n} \nabla w_k(z) \cdot \nabla \xi(z) dz - \int_{\mathbb{R}^n} f'(w_k(z)) \xi(z) dz \\
&= \int_{\mathbb{R}^n} \left[\nabla [u_k(\exp_{x_k}(\epsilon_k z)) \chi_{\frac{R}{\epsilon_k}}(|z|)] \cdot \nabla \xi(z) - f'(u_k(\exp_{x_k}(\epsilon_k z)) \chi_{\frac{R}{\epsilon_k}}(|z|)) \xi(z) \right] dz \\
&= \int_{\mathbb{R}^n} \left[\nabla [u_k(\exp_{x_k}(\epsilon_k z))] \cdot \nabla \xi(z) - f'(u_k(\exp_{x_k}(\epsilon_k z))) \xi(z) \right] dz,
\end{aligned}$$

where in the last equality we have used the fact that for k sufficiently large for any z in the support of ξ $\chi_{\frac{R}{\epsilon_k}}(|z|) = 1$. Now we define the function $\tilde{\xi}_k$ in $H_2^1(M)$ as follows:

$$\tilde{\xi}_k(x) = \begin{cases} \xi\left(\frac{\exp_{x_k}^{-1}(x)}{\epsilon_k}\right) & \forall x \in B_g(x_k, R), \\ 0 & \text{otherwise.} \end{cases}$$

Then we want to write

$$J'(w_k)(\xi) = \frac{\epsilon_k^2}{\epsilon_k^n} \int_M g_{x_k} \left(\nabla u_k(x), \nabla \tilde{\xi}_k(x) \right) d\mu_g - \frac{1}{\epsilon_k^n} \int_M f'(u_k(x)) \tilde{\xi}_k(x) d\mu_g + E_k,$$

where E_k is an error. By hypothesis

$$\begin{aligned} & \left| \int_M \left[\frac{\epsilon_k^2}{\epsilon_k^n} g_{x_k} \left(\nabla u_k(x), \nabla \tilde{\xi}_k(x) \right) - \frac{1}{\epsilon_k^n} f'(u_k(x)) \tilde{\xi}_k(x) \right] d\mu_g \right| \\ &= \left| J'_{\epsilon_k}(u_k)(\tilde{\xi}_k) \right| = o\left(\frac{\epsilon_k}{\epsilon_k^{\frac{n}{2}}} \|\tilde{\xi}\|_{H_2^1(M)} \right) = o(\|\xi\|_{H_2^1(\mathbb{R}^n)}). \end{aligned}$$

Now we have to check the error:

$$\begin{aligned} |E_k| &= \left| \int_{\mathbb{R}^n} [\nabla [u_k(\exp_{x_k}(\epsilon_k z))] \cdot \nabla \xi(z) - f'(u_k(\exp_{x_k}(\epsilon_k z))) \xi(z)] dz \right. \\ &\quad \left. - \frac{\epsilon_k^2}{\epsilon_k^n} \int_M g_{x_k} \left(\nabla u_k(x), \nabla \tilde{\xi}_k(x) \right) d\mu_g - \frac{1}{\epsilon_k^n} \int_M f'(u_k(x)) \tilde{\xi}_k(x) d\mu_g \right| \\ &\leq \left| \int_{\mathbb{R}^n} \nabla [u_k(\exp_{x_k}(\epsilon_k z))] \cdot \nabla \xi(z) dz - \frac{\epsilon_k^2}{\epsilon_k^n} \int_M g_{x_k} \left(\nabla u_k(x), \nabla \tilde{\xi}_k(x) \right) d\mu_g \right| \\ &\quad + \left| \int_{\mathbb{R}^n} f'(u_k(\exp_{x_k}(\epsilon_k z))) \xi(z) dz - \frac{1}{\epsilon_k^n} \int_M f'(u_k(x)) \tilde{\xi}_k(x) d\mu_g \right| \\ &= |E_{1,k}| + |E_{2,k}|. \end{aligned}$$

For the first term we have

$$|E_{1,k}| \leq \int_{\Xi} \left| (\delta^{ij} - g_{x_k}^{ij}(\epsilon_k z)) |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}} \frac{\partial [u_k(\exp_{x_k}(\epsilon_k z))]}{\partial z_i} \frac{\partial \xi(z)}{\partial z_j} \right| dz,$$

where Ξ denotes the compact support of ξ . The limit

$$\lim_{k \rightarrow \infty} |\delta^{ij} - g_{x_k}^{ij}(\epsilon_k z)| |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}} = 0$$

is uniform with respect to $z \in \Xi$. Since

$$\int_{\Xi} \left| \frac{\partial [u_k(\exp_{x_k}(\epsilon_k z))]}{\partial z_i} \frac{\partial \xi(z)}{\partial z_j} \right| dz \leq \|u_k(\exp_{x_k}(\epsilon_k z))\|_{\mathcal{D}^{1,2}(\Xi)} \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}$$

and for k sufficiently large

$$\begin{aligned} \int_{\Xi} |\nabla [u_k(\exp_{x_k}(\epsilon_k z))]|^2 dz &\leq \frac{H}{h^{\frac{n}{2}}} \frac{\epsilon_k^2}{\epsilon_k^n} \int_M |\nabla u_k|_g^2 d\mu_g \\ &\leq \frac{2\mu H}{(\mu - 2)h^{\frac{n}{2}}} J_{\epsilon_k}(u_k) \leq \frac{4\mu H m(J)}{(\mu - 2)h^{\frac{n}{2}}}, \end{aligned}$$

we conclude that $|E_{1,k}|$ tends to zero. For the second term we have

$$|E_{2,k}| = \left| \int_{\Xi} \left(1 - |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}\right) f'(u_k(\exp_{x_k}(\epsilon_k z))) \xi(z) dz \right|.$$

As before, $\lim_{k \rightarrow \infty} |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}$ is 1 uniformly with respect to $z \in \Xi$ and

$$\begin{aligned} & \int_{\Xi} |f'(u_k(\exp_{x_k}(\epsilon_k z))) \xi(z)| dz \\ & \leq \left(\int_{\{z \in \Xi \mid |u_k(\exp_{x_k}(\epsilon_k z))| \geq 1\}} |f'(u_k(\exp_{x_k}(\epsilon_k z)))|^{\frac{p}{p-1}} dz \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} \\ & \quad + \left(\int_{\{z \in \Xi \mid |u_k(\exp_{x_k}(\epsilon_k z))| \leq 1\}} |f'(u_k(\exp_{x_k}(\epsilon_k z)))|^{\frac{q}{q-1}} dz \right)^{\frac{q-1}{q}} \|\xi\|_{L^q(\mathbb{R}^n)}. \end{aligned}$$

It is easy to see that there exists a positive constant C such that the right side is bounded from above by

$$\begin{aligned} & C \left[\left(\frac{1}{\epsilon_k^n} \int_M f'(u_k) u_k d\mu_g \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} + \left(\frac{1}{\epsilon_k^n} \int_M f'(u_k) u_k d\mu_g \right)^{\frac{q-1}{q}} \|\xi\|_{L^q(\mathbb{R}^n)} \right] \\ & \leq C \left[\left(\frac{2\mu}{\mu-2} (m(J) + 1) \right)^{\frac{p-1}{p}} \|\xi\|_{L^p(\mathbb{R}^n)} + \left(\frac{2\mu}{\mu-2} (m(J) + 1) \right)^{\frac{q-1}{q}} \|\xi\|_{L^q(\mathbb{R}^n)} \right] \end{aligned}$$

and this proves that $|E_{2,k}|$ tends to zero. Our second and last step is to prove that for any $\xi \in C_0^\infty(\mathbb{R}^n)$ $J'(w_k)(\xi)$ tends to $J'(w)(\xi)$ for k tending to infinity. It is immediate that $\int_{\mathbb{R}^n} \nabla w_k \cdot \nabla \xi dz$ tends to $\int_{\mathbb{R}^n} \nabla w \cdot \nabla \xi dz$. By mean value theorem there exists a function $\theta(z)$ with values in $(0, 1)$ such that

$$\begin{aligned} & \int_{\mathbb{R}^n} |f'(w_k(z)) - f'(w(z))| |\xi(z)| dz \\ & = \int_{\mathbb{R}^n} |f''(\theta(z)w_k(z) + (1-\theta(z))w(z))| |w_k(z) - w(z)| |\xi(z)| dz. \end{aligned}$$

By Hölder inequality the righthand side is bounded from above by

$$\|w_k - w\|_{L^p(\Xi)} \|\xi\|_{L^p(\Xi)} \left(\int_{\mathbb{R}^n} |f''(\theta(z)w_k(z) + (1-\theta(z))w(z))|^{\frac{p}{p-2}} dz \right)^{\frac{p-2}{p}},$$

where $\|w_k - w\|_{L^p(\Xi)}$ tends to zero by (i). Besides we have

$$\begin{aligned} & \int_{\mathbb{R}^n} |f''(\theta(z)w_k(z) + (1-\theta(z))w(z))|^{\frac{p}{p-2}} dz \\ & \leq c_1 \int_{\{z \in \Xi \mid |\theta(z)w_k(z) + (1-\theta(z))w(z)| \geq 1\}} |\theta(z)w_k(z) + (1-\theta(z))w(z)|^p dz + c_1 \text{vol}(\Xi) \\ & \leq c_1 2^{p-1} (\|w_k\|_{L^p(\Xi)}^p + \|w\|_{L^p(\Xi)}^p) + c_1 \text{vol}(\Xi) \end{aligned}$$

and this quantity is bounded by a constant.

(iii) Let $t_k = t(w_k)$ be the multiplier defined in (ii), Lemma 4.1. First of all we prove that there exist $0 < t_1 \leq 1 \leq t_2$ such that for all k $t_1 \leq t_k \leq t_2$. Let $g_w(t) = J(tw)$. By Lemma 4.1 (ii), it is enough to find $t_1 > 0$ such that for all $t \in [0, t_1]$ $g'_{w_k}(t) > 0$ for all $k \in \mathbb{N}$. There holds

$$\begin{aligned} g'_{w_k}(t) &= t \int_{\mathbb{R}^n} |\nabla w_k(z)|^2 dz - \int_{\mathbb{R}^n} f'(tw_k(z)) w_k(z) dz \\ &> t \int_{\mathbb{R}^n} |\nabla w_k(z)|^2 dz - \frac{c_1 t^{p-1}}{c_0} \int_{\mathbb{R}^n} f(w_k(z)) dz. \end{aligned}$$

Since we have

$$\begin{aligned} \int_{\mathbb{R}^n} |\nabla w_k(z)|^2 dz &\geq \frac{h \epsilon_k^2}{H^{\frac{n}{2}} \epsilon_k^n} \int_{B_g(x_k, \frac{R}{2})} |\nabla u_k|_g^2 d\mu_g \\ &\geq \frac{2h}{H^{\frac{n}{2}}} \left(\frac{\epsilon_k^2}{2\epsilon_k^n} \int_{B_g(x_k, \frac{R}{2})} |\nabla u_k|_g^2 d\mu_g - \frac{1}{\epsilon_k^n} \int_{B_g(x_k, \frac{R}{2})} f(u_k) d\mu_g \right) \geq \frac{2h}{H^{\frac{n}{2}}} \gamma, \end{aligned}$$

where we have used the second equation of (5.9), and

$$\begin{aligned} \int_{\mathbb{R}^n} f(w_k(z)) dz &\leq \frac{1}{h^{\frac{n}{2}} \epsilon_k^n} \int_{B_g(x_k, \frac{R}{2})} f(u_k) d\mu_g \\ &\leq \frac{2}{h^{\frac{n}{2}} (\mu - 2) \epsilon_k^n} \int_{B_g(x_k, \frac{R}{2})} \left[\frac{1}{2} f'(u_k) u_k - f(u_k) \right] d\mu_g \leq \frac{2(m(J) + 1)}{h^{\frac{n}{2}} (\mu - 2)}, \end{aligned}$$

then there exist $C_1, C_2 > 0$ such that $g'_{w_k}(t) > C_1 t - C_2 t^{p-1}$. So we consider $t_1 = \left(\frac{C_1}{C_2} \right)^{\frac{1}{p-2}}$.

If v is a function in the Nehari manifold \mathcal{N} , $J(v) = \tilde{F}_{\mathbb{R}^n}(v)$, as defined in (2.10). Then by property (f1) $J(v)$ is positive. By Lemma 4.1 (ii), it is enough to find $t_2 > 0$ such that for all $t \geq t_2$ $J(tw_k) < 0$ for all $k \in \mathbb{N}$. Since

$$J(tw_k) = \frac{t^2}{2} \int_{\mathbb{R}^n} |\nabla w_k(z)|^2 dz - \int_{\mathbb{R}^n} f(tw_k(z)) dz$$

and we already proved that $\{w_k\}_{k \in \mathbb{N}}$ is bounded in $\mathcal{D}^{1,2}(\mathbb{R}^n)$, we still have to bound the second part for $t \geq 1$

$$\begin{aligned} \int_{\mathbb{R}^n} f(tw_k(z)) dz &\geq c_0 t^p \left(\int_{\{z \in \mathbb{R}^n \mid |w_k(z)| \geq 1\}} |w_k(z)|^p dz + \int_{\{z \in \mathbb{R}^n \mid |w_k(z)| \leq 1\}} |w_k(z)|^q dz \right) \\ &> \frac{c_0 t^p}{c_1} \int_{\mathbb{R}^n} f''(w_k(z)) (w_k(z))^2 dz > \frac{2c_0 t^p}{c_1 - 2c_0} \tilde{F}_{\mathbb{R}^n}(w_k) \\ &\geq \frac{2c_0 t^p}{(c_1 - 2c_0) H^{\frac{n}{2}}} \tilde{F}_{\epsilon_k, B_g(x_k, \frac{R}{2})}(u_k) \geq \frac{2c_0 \gamma t^p}{(c_1 - 2c_0) H^{\frac{n}{2}}}, \end{aligned}$$

where we have used (5.9). So there exist $C_3, C_4 > 0$ such that $J(tw_k) < C_3 t^2 - C_4 t^p$ and $t_2 = \left(\frac{C_3}{C_4}\right)^{\frac{1}{p-2}}$.

By the boundedness of t_k we conclude that up to subsequences t_k converges to \bar{t} for k tending to infinity.

We apply the splitting lemma (Lemma 5.5) to the sequence $t_k w_k$. Then in the first case we have that $t_k w_k$ converges strongly in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ to a ground state solution \bar{w} . It is easy to see that $t_k w_k$ weakly converges to $\bar{t}w$, in fact for any $\xi \in C_0^\infty(\mathbb{R}^n)$ there holds

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \nabla(t_k w_k - \bar{t}w) \cdot \nabla \xi \right| &= \left| \int_{\mathbb{R}^n} \nabla(t_k w_k - \bar{t}w_k) \cdot \nabla \xi + \int_{\mathbb{R}^n} \nabla(\bar{t}w_k - \bar{t}w) \cdot \nabla \xi \right| \\ &\leq |t_k - \bar{t}| \|\xi\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} \|w_k\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)} + o(1) = o(1). \end{aligned}$$

We can conclude that $\bar{w} = \bar{t}w$. In particular $w \not\equiv 0$ and by the fact that both \bar{w} and w are in \mathcal{N} , $\bar{t} = 1$ and we have finished.

Otherwise, there exist a sequence of points $\{y_k\}_{k \in \mathbb{N}}$ tending to infinity, a ground state solution U and a sequence of functions $\{w_k^0\}_{k \in \mathbb{N}}$ such that, up to a subsequence $t_k w_k(z) = w_k^0(z) + U(z - y_k)$ for all $z \in \mathbb{R}^n$ and w_k^0 tends strongly to zero. We consider three different cases: $\lim_{k \rightarrow \infty} |y_k| - \frac{R}{\epsilon_k} = 2T > 0$, $\lim_{k \rightarrow \infty} |y_k| - \frac{R}{\epsilon_k} = 0$ and $\lim_{k \rightarrow \infty} \frac{R}{\epsilon_k} - |y_k| = 2T > 0$. In the first case, since by definition $w_k \equiv 0$ in $\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)$, $w_k^0(z) = -U(z - y_k)$. Then we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)} |\nabla w_k^0(z)|^2 dz &= \int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)} |\nabla U(z - y_k)|^2 dz \\ &\geq \int_{B(y_k, T)} |\nabla U(z - y_k)|^2 dz = \int_{B(0, T)} |\nabla U(z)|^2 dz > 0 \end{aligned}$$

and this is in contradiction with the fact that w_k^0 tends strongly to zero. If $\lim_{k \rightarrow \infty} |y_k| - \frac{R}{\epsilon_k} = 0$, let $\pi(y_k)$ denote the projection of y_k onto the sphere centred in the origin with radius $\frac{R}{\epsilon_k}$ and $T > 0$. Then

$$\begin{aligned} \int_{\{z \in B(\pi(y_k), T) \mid |z| \geq \frac{R}{\epsilon_k}\}} |\nabla U(z - \pi(y_k))|^2 dz &= \int_{\{z \in B(0, T) \mid |z + \pi(y_k)| \geq \frac{R}{\epsilon_k}\}} |\nabla U(z)|^2 dz \\ &\geq \min_{\zeta \in S^n} \int_{\{z \in B(0, T) \mid z \cdot \zeta \geq 0\}} |\nabla U(z)|^2 dz = C > 0 \end{aligned}$$

where S^n is the unit sphere in \mathbb{R}^n and $z \cdot \zeta$ is the scalar product in \mathbb{R}^n . Similarly to the first case we have

$$\begin{aligned} \int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)} |\nabla w_k^0(z)|^2 dz &= \int_{\mathbb{R}^n \setminus B\left(0, \frac{R}{\epsilon_k}\right)} |\nabla U(z - y_k)|^2 dz \\ &\geq \int_{\{z \in B(y_k, T) \mid |z| \geq \frac{R}{\epsilon_k}\}} |\nabla U(z - y_k)|^2 dz \\ &= \int_{\{z \in B(\pi(y_k), T) \mid |z| \geq \frac{R}{\epsilon_k}\}} |\nabla U(z - \pi(y_k))|^2 dz + o(1) \end{aligned}$$

and this is greater than $\frac{C}{2}$ for k sufficiently large, which is a contradiction. Finally, if $\lim_{k \rightarrow \infty} \frac{R}{\epsilon_k} - |y_k| = 2T > 0$, for k sufficiently large $B(y_k, T)$ is contained in $B\left(0, \frac{R}{\epsilon_k}\right)$. There holds

$$\begin{aligned} & \int_{B(y_k, T)} \left[\frac{1}{2} f'(U(z - y_k)) U(z - y_k) - f(U(z - y_k)) \right] dz \\ &= \int_{B(0, T)} \left[\frac{1}{2} f'(U(z)) U(z) - f(U(z)) \right] dz = \gamma_0 > 0. \end{aligned}$$

We consider the new sequence of points

$$\tilde{x}_k = \exp_{x_k}(\epsilon_k y_k) \in B_g(x_k, R).$$

For any k sufficiently large, let $U(\tilde{x}_k)$ be the neighborhood of \tilde{x}_k defined as $\exp_{x_k}(\epsilon_k B(y_k, T))$, then

$$\begin{aligned} & \frac{1}{\epsilon_k^n} \int_{U(\tilde{x}_k)} \left[\frac{1}{2} f'(u_k) u_k - f(u_k) \right] d\mu_g \\ &= \frac{1}{\epsilon_k^n} \int_{\epsilon_k B(y_k, T)} \left[\frac{1}{2} f'(u_k(\exp_{x_k}(z))) u_k(\exp_{x_k}(z)) - f(u_k(\exp_{x_k}(z))) \right] |g_{x_k}(z)|^{\frac{1}{2}} dz \\ &\geq h^{\frac{n}{2}} \int_{B(y_k, T)} \left[\frac{1}{2} f'(w_k(z)) w_k(z) - f(w_k(z)) \right] dz. \end{aligned}$$

Since $t_k \in (t_1, t_2)$ and using the properties of the function f we obtain

$$\begin{aligned} & \int_{B(y_k, T)} \left[\frac{1}{2} f'(w_k(z)) w_k(z) - f(w_k(z)) \right] dz \\ &\geq \int_{B(y_k, T)} \left[\frac{1}{2} f' \left(\frac{t_k}{t_2} w_k(z) \right) \frac{t_k}{t_2} w_k(z) - f \left(\frac{t_k}{t_2} w_k(z) \right) \right] dz \\ &> \frac{(\mu - 2)c_0}{(c_1 - 2c_0)t_2^q} \int_{B(y_k, T)} \left[\frac{1}{2} f'(t_k w_k(z)) t_k w_k(z) - f(t_k w_k(z)) \right] dz. \end{aligned}$$

By the splitting lemma we have

$$\begin{aligned} & \int_{B(y_k, T)} \left[\frac{1}{2} f'(t_k w_k(z)) t_k w_k(z) - f(t_k w_k(z)) \right] dz \\ &= \int_{B(y_k, T)} \left[\frac{1}{2} f'(w_k^0(z) + U(z - y_k)) (w_k^0(z) + U(z - y_k)) - f(w_k^0(z) + U(z - y_k)) \right] dz \\ &= \int_{B(y_k, T)} \left[\frac{1}{2} f'(U(z - y_k)) (U(z - y_k)) - f(U(z - y_k)) \right] dz + o(1) \\ &= \gamma_0 + o(1). \end{aligned}$$

So we have proved that for any k sufficiently large

$$\frac{1}{\epsilon_k^n} \int_{U(\tilde{x}_k)} \left[\frac{1}{2} f'(u_k) u_k - f(u_k) \right] d\mu_g > \tilde{\gamma}_0 > 0. \quad (5.13)$$

By definition, for k big enough $U(\tilde{x}_k)$ is contained in $B_g(\tilde{x}_k, R)$ and so we can substitute x_k by \tilde{x}_k and w_k by \tilde{w}_k , defined as in (5.10) with the new choice of points. Steps (i) and (ii) are independent of x_k (provided w_k is not identically zero) and so \tilde{w}_k tends weakly to a weak solution \tilde{w} . It is possible to see that there exists $\tilde{T} > 0$ such that for any k $U(\tilde{x}_k) \subset B_g(\tilde{x}_k, \epsilon_k \tilde{T})$. Then we have

$$\begin{aligned} & \int_{B(0, \tilde{T})} \left[\frac{1}{2} f'(\tilde{w}_k(z)) \tilde{w}_k(z) - f(\tilde{w}_k(z)) \right] dz \\ & \geq \frac{1}{H^{\frac{n}{2}} \epsilon_k^n} \int_{B_g(\tilde{x}_k, \epsilon_k \tilde{T})} \left[\frac{1}{2} f'(u_k(x)) u_k(x) - f(u_k(x)) \right] d\mu_g \\ & \geq \frac{1}{H^{\frac{n}{2}} \epsilon_k^n} \int_{U(\tilde{x}_k)} \left[\frac{1}{2} f'(u_k(x)) u_k(x) - f(u_k(x)) \right] d\mu_g. \end{aligned}$$

By (5.13) and by the strong convergence of \tilde{w}_k to \tilde{w} in $L^p(B(0, \tilde{T}))$, we conclude that

$$\int_{B(0, \tilde{T})} \left[\frac{1}{2} f'(\tilde{w}(z)) \tilde{w}(z) - f(\tilde{w}(z)) \right] dz \geq \frac{\tilde{\gamma}_0}{H^{\frac{n}{2}}}$$

and so $\tilde{w} \not\equiv 0$ and $\tilde{w} \in \mathcal{N}$.

From now on we will write as before w_k instead of \tilde{w}_k , x_k instead of \tilde{x}_k and w instead of \tilde{w} . The last step is to verify that $J(w) = m(J)$. Let us consider the following inequalities

$$\begin{aligned} m(J) + \delta_k & \geq J_{\epsilon_k}(u_k) = \frac{1}{\epsilon_k^n} \int_M \left[\frac{1}{2} f'(u_k) u_k - f(u_k) \right] d\mu_g \\ & \geq \int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w_k) w_k - f(w_k) \right] |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}} dz. \end{aligned} \tag{5.14}$$

We define the sequence of functions in $L^2(\mathbb{R}^n)$:

$$F_k(z) = \left[\frac{1}{2} f'(w_k(z)) w_k(z) - f(w_k(z)) \right]^{\frac{1}{2}} |g_{x_k}(\epsilon_k z)|^{\frac{1}{4}}.$$

By (5.14) this sequence is bounded in $L^2(\mathbb{R}^n)$ and there exists a weak limit $F \in L^2(\mathbb{R}^n)$. We prove that

$$F(z) = \left[\frac{1}{2} f'(w(z)) w(z) - f(w(z)) \right]^{\frac{1}{2}}. \tag{5.15}$$

Let ξ be in $C_0^\infty(\mathbb{R}^n)$. On Ξ , the support of ξ , w_k strongly converges to w in $L^p(\Xi)$. So up to a subsequence $w_k(z)$ converges to $w(z)$ almost everywhere. Then pointwise

$$F_k(z) \xi(z) \rightarrow \left[\frac{1}{2} f'(w(z)) w(z) - f(w(z)) \right]^{\frac{1}{2}} \xi(z)$$

for almost every $z \in \Xi$. We can now apply Lebesgue theorem. In fact, there holds

$$\begin{aligned} |F_k(z)| |\xi(z)| &< \begin{cases} H^{\frac{n}{4}} \left(\frac{c_1}{2} - c_0\right)^{\frac{1}{2}} |w_k(z)|^{\frac{p}{2}} |\xi(z)| & \text{if } |w_k(z)| \geq 1 \\ H^{\frac{n}{4}} \left(\frac{c_1}{2} - c_0\right)^{\frac{1}{2}} |w_k(z)|^{\frac{p}{2}} |\xi(z)| & \text{if } |w_k(z)| \leq 1 \end{cases} \\ &\leq H^{\frac{n}{4}} \left(\frac{c_1}{2} - c_0\right)^{\frac{1}{2}} (1 + |w_k(z)|^{\frac{p}{2}}) |\xi(z)| \end{aligned}$$

and, since w_k converges strongly to w in $L^p(\Xi)$, there exists $W \in L^p(\Xi)$ such that for all k $|w_k(z)| \leq W(z)$ almost everywhere and $|F_k(z)| |\xi(z)| \leq H^{\frac{n}{4}} \left(\frac{c_1}{2} - c_0\right)^{\frac{1}{2}} (1 + (W(z))^{\frac{p}{2}}) |\xi(z)| \in L^2(\Xi)$. So (5.15) is proved. By weak lower semicontinuity of the norm

$$\|F\|_{L^2(\mathbb{R}^n)}^2 \leq \liminf_{k \rightarrow \infty} \|F_k\|_{L^2(\mathbb{R}^n)}^2,$$

that is

$$\int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w) w - f(w) \right] dz \leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w_k) w_k - f(w_k) \right] |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}} dz.$$

By this inequality and (5.14) we conclude that

$$\begin{aligned} m(J) &= \lim_{k \rightarrow \infty} m(J) + \delta_k \geq \lim_{k \rightarrow \infty} J_{\epsilon_k}(u_k) \\ &\geq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w_k) w_k - f(w_k) \right] |g_{x_k}(\epsilon_k z)|^{\frac{1}{2}} dz \\ &\geq \int_{\mathbb{R}^n} \left[\frac{1}{2} f'(w) w - f(w) \right] dz \geq m(J). \end{aligned}$$

(iv) The equality is immediate from (5.14). \square

We recall here Ekeland Principle (see for instance [12]).

Definition 5.7. Let X be a complete metric space and $\Psi : X \rightarrow \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function on X , bounded from below. Given $\eta > 0$ and $\bar{u} \in X$ such that

$$\Psi(\bar{u}) < \inf_{u \in X} \Psi(u) + \frac{\eta}{2},$$

for all $\lambda > 0$ there exists $u_\lambda \in X$ such that

$$\Psi(u_\lambda) < \Psi(\bar{u}), \quad d(u_\lambda, \bar{u}) < \lambda$$

and for all $u \neq u_\lambda$ it holds

$$\Psi(u_\lambda) < \Psi(u) + \frac{\eta}{\lambda} d(u_\lambda, u).$$

Remark 5.8. 1. We apply Lemma 5.6 when u_k is a minimum solution $u_k \in \mathcal{N}_{\epsilon_k}$, $J_{\epsilon_k}(u_k) = m_{\epsilon_k}$. By (iv) we have $\lim_{k \rightarrow \infty} m_{\epsilon_k} = m(J)$. In particular for any $\delta > 0$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ sufficiently small such that for all $\epsilon \leq \epsilon_0$ $|m_\epsilon - m(J)| < \delta$.

2. Applying Ekeland principle for $X = \Sigma_{\epsilon, \delta}$, with $\epsilon \leq \epsilon_0(\delta)$ as in 1, we obtain that for all $\bar{u} \in \Sigma_{\epsilon, \delta}$ there exists $u_\delta \in \Sigma_{\epsilon, \delta}$ such that

$$J_\epsilon(u_\delta) < J_\epsilon(\bar{u}), \quad \frac{\epsilon}{\epsilon^{\frac{n}{2}}} \|u_\delta - \bar{u}\|_{H_2^1(M)} < 4\sqrt{\delta}$$

and for all $u \in T\Sigma_{\epsilon, \delta}$

$$|J'_\epsilon(u_\delta)(u)| < \frac{\sqrt{\delta}\epsilon}{\epsilon^{\frac{n}{2}}} \|u\|_{H_2^1(M)}. \quad (5.16)$$

Proof of Proposition 5.4. We choose $\epsilon_0(\delta)$ as in point 1 of Remark 5.8. We also assume that $\epsilon_0(\delta_0)$ is less than ϵ_0 in Lemma 5.1.

By contradiction we assume that there is $\eta_0 \in (0, 1)$ such that there exist two positive sequences $\{\delta_k\}_{k \in \mathbb{N}}$, $\{\epsilon_k\}_{k \in \mathbb{N}}$ tending to zero as k tends to infinity and a sequence of functions $\{u_k\}_{k \in \mathbb{N}}$, with $u_k \in \Sigma_{\epsilon_k, \delta_k}$, and for any $x \in M$

$$\tilde{F}_{\epsilon_k, B_g(x, \frac{r(M)}{2})}(u_k) \leq \eta_0 m(J). \quad (5.17)$$

By Ekeland principle for any k we can consider \tilde{u}_k as in 2 of Remark 5.8. Property (5.17) becomes

$$\tilde{F}_{\epsilon_k, B_g(x, \frac{r(M)}{2})}(\tilde{u}_k) \leq \eta_1 m(J) \quad (5.18)$$

with η_1 still in $(0, 1)$. To prove this we have to evaluate the difference

$$\frac{1}{\epsilon_k^n} \int_{B_g(x, \frac{r(M)}{2})} \left| \frac{1}{2} f'(\tilde{u}_k) \tilde{u}_k - f(\tilde{u}_k) - \frac{1}{2} f'(u_k) u_k + f(u_k) \right| d\mu_g,$$

which by mean value theorem can be written

$$\frac{1}{2\epsilon_k^n} \int_B |f''(u_k^*) u_k^* - f'(u_k^*)| |\tilde{u}_k - u_k| d\mu_g, \quad (5.19)$$

where B is $B_g(x, \frac{r(M)}{2})$ and $u_k^*(x) = \theta(x) \tilde{u}_k(x) + (1 - \theta(x)) u_k(x)$ for a suitable function $\theta(x)$ with values in $(0, 1)$. By Hölder inequality (5.19) is bounded from above by

$$\frac{1}{2} \left(\frac{1}{\epsilon_k^n} \int_B |f''(u_k^*) u_k^* - f'(u_k^*)|^{\frac{2n}{n+2}} d\mu_g \right)^{\frac{n+2}{2n}} \left(\frac{1}{\epsilon_k^n} \int_B |\tilde{u}_k - u_k|^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2n}}.$$

We prove that the first factor is bounded and the second one is infinitesimal. In fact, we have

$$\begin{aligned} \left(\frac{1}{\epsilon_k^n} \int_B |\tilde{u}_k - u_k|^{\frac{2n}{n-2}} d\mu_g \right)^{\frac{n-2}{2n}} &= \frac{\epsilon_k}{\epsilon_k^{\frac{n}{2}}} \|\tilde{u}_k - u_k\|_{L^{\frac{2n}{n-2}}(B)} \\ &\leq C \frac{\epsilon_k}{\epsilon_k^{\frac{n}{2}}} \|\tilde{u}_k - u_k\|_{H_2^1(M)} < 4C\sqrt{\delta}. \end{aligned}$$

The proof of the bound

$$\frac{1}{\epsilon_k^n} \int_B |f''(u_k^*)u_k^* - f'(u_k^*)|^{\frac{2n}{n+2}} d\mu_g \leq C \quad (5.20)$$

for a positive constant C can be found in the Appendix.

We apply Lemma 5.6 to the sequences $\{\delta_k\}_{k \in \mathbb{N}}$, $\{\epsilon_k\}_{k \in \mathbb{N}}$ and $\{\tilde{u}_k\}_{k \in \mathbb{N}}$, obtaining a sequence of functions on \mathbb{R}^n $\{w_k\}_{k \in \mathbb{N}}$ (it is easy to see that (5.16) holds for any $u \in H_2^1(M)$). Let w be the weak limit in $\mathcal{D}^{1,2}(\mathbb{R}^n)$ of w_k . Let η_2 be a constant in $(0, 1)$ such that $\eta_2 > \frac{1+\eta_1}{2}$. Since $J(w) = m(J)$, there exists $T > 0$ such that

$$\int_{B(0,T)} \left[\frac{1}{2} f'(w(z))w(z) - f(w(z)) \right] dz \geq \eta_2 m(J). \quad (5.21)$$

On the other hand, up to a subsequence, we have

$$\begin{aligned} \int_{B(0,T)} \left[\frac{1}{2} f'(w)w - f(w) \right] dz &= \lim_{k \rightarrow \infty} \int_{B(0,T)} \left[\frac{1}{2} f'(w_k)w_k - f(w_k) \right] dz \\ &= \lim_{k \rightarrow \infty} \frac{1}{\epsilon_k} \int_{B(0, \epsilon_k T)} \left[\frac{1}{2} f'(\tilde{u}_k \circ \exp_{x_k}) \tilde{u}_k \circ \exp_{x_k} - f(\tilde{u}_k \circ \exp_{x_k}) \right] dz. \end{aligned} \quad (5.22)$$

By compactness the sequence x_k converges (up to a subsequence) to \bar{x} and for any $z \in B(0, T)$ the limit of $|g_{x_k}(\epsilon_k z)|^{\frac{1}{2}}$ for k tending to infinity is $|g_{\bar{x}}(0)|^{\frac{1}{2}} = 1$. Since $\frac{2\eta_1}{1+\eta_1} \in (0, 1)$, for k sufficiently big for any $z \in B(0, \epsilon_k T)$ we have $|g_{x_k}(z)|^{\frac{1}{2}} > \frac{2\eta_1}{1+\eta_1}$. So the last limit in (5.22) is less than

$$\begin{aligned} \frac{1+\eta_1}{2\eta_1} \lim_{k \rightarrow \infty} \frac{1}{\epsilon_k} \int_{B(0, \epsilon_k T)} \left[\frac{1}{2} f'(\tilde{u}_k \circ \exp_{x_k}) \tilde{u}_k \circ \exp_{x_k} - f(\tilde{u}_k \circ \exp_{x_k}) \right] |g_{x_k}(z)|^{\frac{1}{2}} dz \\ = \frac{1+\eta_1}{2\eta_1} \lim_{k \rightarrow \infty} \frac{1}{\epsilon_k} \int_{B(x_k, \epsilon_k T)} \left[\frac{1}{2} f'(\tilde{u}_k) \tilde{u}_k - f(\tilde{u}_k) \right] d\mu_g \leq \frac{1+\eta_1}{2} m(J), \end{aligned}$$

where we have used property (5.18). By this inequality together with (5.22) and (5.21) we get $\eta_2 \leq \frac{1+\eta_1}{2}$ which is in contradiction with the choice of η_2 . \square

It is now possible to prove the following proposition:

Proposition 5.9. *There exists $\delta_0 \in (0, m(J))$ such that for any $\delta \in (0, \delta_0)$ there exists $\epsilon_0 = \epsilon_0(\delta) > 0$ and for any $\epsilon \in (0, \epsilon_0)$ and $u \in \Sigma_{\epsilon, \delta}$ the barycentre $\beta(u)$ is in $M_r(M)$.*

Proof. By Proposition 5.4, for any $\eta \in (0, 1)$ and for any $u \in \Sigma_{\epsilon, \delta}$ with ϵ and δ sufficiently small there exists a point x_0 such that

$$\tilde{F}_{\epsilon, B_g(x_0, \frac{r(M)}{2})}(u) > \eta m(J).$$

Since $u \in \Sigma_{\epsilon, \delta}$ we also have

$$\tilde{F}_{\epsilon, M}(u) \leq m(J) + \delta.$$

We define

$$\rho(u(x)) = \frac{\frac{1}{2}f'(u(x))u(x) - f(u(x))}{\int_M [\frac{1}{2}f'(u(x))u(x) - f(u(x))] d\mu_g}.$$

By the previous inequalities we have then

$$\int_{B_g(x_0, \frac{r(M)}{2})} \rho(u(x)) d\mu_g > \frac{\eta}{1 + \frac{\delta}{m(J)}}.$$

We can now esteem

$$\begin{aligned} |\beta(u) - x_0| &= \left| \int_M (x - x_0) \rho(u(x)) d\mu_g \right| \\ &\leq \left| \int_{B_g(x_0, \frac{r(M)}{2})} (x - x_0) \rho(u(x)) d\mu_g \right| + \left| \int_{M \setminus B_g(x_0, \frac{r(M)}{2})} (x - x_0) \rho(u(x)) d\mu_g \right| \\ &< \frac{r(M)}{2} + D \left(1 - \frac{\eta}{1 + \frac{\delta}{m(J)}} \right), \end{aligned}$$

where D is the diameter of the manifold M . For η near to 1 and δ sufficiently small we obtain $\beta(u) \in M_{r(M)}$. \square

6 The function I_ϵ

We prove now that the composition I_ϵ of ϕ_ϵ and β is well defined and homotopic to the identity on M :

Proposition 6.1. *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$ the composition*

$$I_\epsilon = \beta \circ \phi_\epsilon : M \rightarrow M_{r(M)}$$

is well defined and homotopic to the identity on M .

Proof. Let us consider the function $H : [0, 1] \times M \rightarrow M_{r(M)}$, defined by $H(t, x) = tI_\epsilon(x) + (1-t)x$. This function is a homotopy if for any $t \in [0, 1]$ $H(t, x) \in M_{r(M)}$. It is enough to prove that for any $x_0 \in M$ $|I_\epsilon(x_0) - x_0| < r(M)$. Since the support of $\phi_\epsilon(x_0)$ is contained in $B_g(x_0, R)$

$$\begin{aligned} I_\epsilon(x_0) - x_0 &= \int_M (x - x_0) \rho(\phi_\epsilon(x_0)(x)) d\mu_g = \int_{B_g(x_0, R)} (x - x_0) \rho(\phi_\epsilon(x_0)(x)) d\mu_g \\ &= \frac{\int_{B(0, R)} z \Phi(t_\epsilon(W_{x_0, \epsilon})W_{x_0, \epsilon}(\exp_{x_0}(z))) |g_{x_0}(z)|^{\frac{1}{2}} dz}{\int_{B(0, R)} \Phi(t_\epsilon(W_{x_0, \epsilon})W_{x_0, \epsilon}(\exp_{x_0}(z))) |g_{x_0}(z)|^{\frac{1}{2}} dz} \\ &= \frac{\epsilon \int_{B(0, \frac{R}{\epsilon})} z \Phi(t_\epsilon(W_{x_0, \epsilon})W_{x_0, \epsilon}(\exp_{x_0}(\epsilon z))) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz}{\int_{B(0, \frac{R}{\epsilon})} \Phi(t_\epsilon(W_{x_0, \epsilon})W_{x_0, \epsilon}(\exp_{x_0}(\epsilon z))) |g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz}, \end{aligned}$$

where Φ is defined in (5.2). We recall that for any $\epsilon \in (0, 1]$ and $x_0 \in M$ $t_1 \leq t_\epsilon(W_{x_0, \epsilon}) \leq t_2$. By definition of ϕ_ϵ , we have

$$\int_{B(0, \frac{R}{\epsilon})} \Phi(t_\epsilon(W_{x_0, \epsilon})W_{x_0, \epsilon}(\exp_{x_0}(\epsilon z)))|g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz \geq h^{\frac{n}{2}} \int_{B(0, R)} \Phi(t_1(U(z) - \tilde{U}_R)) dz > 0,$$

where \tilde{U}_R is the value $U(z)$ for any $z \in \mathbb{R}^n$ such that $|z| = R$. Furthermore, we have

$$\begin{aligned} & \epsilon \int_{B(0, \frac{R}{\epsilon})} |z| |\Phi(t_\epsilon(W_{x_0, \epsilon})W_{x_0, \epsilon}(\exp_{x_0}(\epsilon z)))| g_{x_0}(\epsilon z)|^{\frac{1}{2}} dz \leq \epsilon H^{\frac{n}{2}} \int_{B(0, \frac{R}{\epsilon})} |z| \Phi(t_2 U(z)) dz \\ & < \frac{(c_1 - 2c_0)H^{\frac{n}{2}}\epsilon}{2} \left[\int_{\{z \in B(0, \frac{R}{\epsilon}) \mid t_2 U(z) \geq 1\}} |z| t_2^p(U(z))^p dz + \int_{\{z \in B(0, \frac{R}{\epsilon}) \mid t_2 U(z) \leq 1\}} |z| t_2^q(U(z))^q dz \right]. \end{aligned}$$

Since U is spherically symmetric and decreasing, there exists $\rho_0 > 0$ such that the last quantity is equal to

$$\frac{(c_1 - 2c_0)H^{\frac{n}{2}}\epsilon}{2} \left[\int_{B(0, \rho_0)} |z| t_2^p(U(z))^p dz + \int_{B(0, \frac{R}{\epsilon}) \setminus B(0, \rho_0)} |z| t_2^q(U(z))^q dz \right]. \quad (6.1)$$

Obviously, the integral

$$\int_{B(0, \rho_0)} |z| t_2^p(U(z))^p dz \leq t_2^p \rho_0 \int_{B(0, \rho_0)} (U(z))^p dz$$

is bounded. For the second integral in (6.1), we use the well known inequality by Strauss (see [16]):

$$\epsilon \int_{B(0, \frac{R}{\epsilon}) \setminus B(0, \rho_0)} |z| (U(z))^q dz \leq C_n \|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^n)}^q \epsilon \int_{B(0, \frac{R}{\epsilon}) \setminus B(0, \rho_0)} \frac{|z|}{|z|^{\frac{(n-2)q}{2}}} dz$$

where C_n is a positive constant. Then we conclude that there exist two positive constants C_1, C_2 such that (6.1) is bounded from above by $C_1 \epsilon + C_2 \epsilon^{\frac{(n-2)q-2n}{2}}$, where the second exponent is positive and so $|I_\epsilon(x_0) - x_0|$ tends to zero as ϵ tends to zero. \square

Finally, by standard arguments it is easy to see that the Palais-Smale condition holds for J_ϵ constrained on \mathcal{N}_ϵ .

7 The Morse theory result

For an introduction to Morse theory we refer the reader to [15], while for the applications to problems of functional analysis we mention [2].

Let (X, Y) be a couple of topological spaces, with $Y \subset X$, and $H_k(X, Y)$ be the k -th homology group with coefficients in some field. We recall the following definition:

Definition 7.1. *The Poincaré polynomial of (X, Y) is the formal power series*

$$\mathcal{P}_t(X, Y) = \sum_{k=0}^{\infty} \dim[H_k(X, Y)] t^k.$$

The Poincaré polynomial of X is defined as $\mathcal{P}_t(X) = \mathcal{P}_t(X, \emptyset)$.

If X is a compact n -dimensional manifold $\dim[H_k(X)]$ is finite for any k and $\dim[H_k(X)] = 0$ for any $k > n$. In particular $\mathcal{P}_t(X)$ is a polynomial and not a formal series.

We define now the Morse index.

Definition 7.2. *Let J be a C^2 functional on a Banach space X and let u be an isolated critical point of J with $J(u) = c$. The (polynomial) Morse index of u is defined as*

$$i_t(u) = \sum_{k=0}^{\infty} \dim[H_k(J^c, J^c \setminus \{u\})] t^k,$$

where $J^c = \{v \in X \mid J(v) \leq c\}$. If u is a non degenerate critical point then $i_t(u) = t^{\mu(u)}$, where $\mu(u)$ is the (numerical) Morse index of u and represents the dimension of the maximal subspace on which the bilinear form $J''(u)[\cdot, \cdot]$ is negative definite.

It is now possible to state Theorem 1.2 more precisely:

Theorem 7.3. *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, if the set K_ϵ of solutions of equation (1.1) is discrete, then*

$$\sum_{u \in K_\epsilon} i_t(u) = t\mathcal{P}_t(M) + t^2[\mathcal{P}_t(M) - 1] + t(1+t)\mathcal{Q}_\epsilon(t),$$

where $\mathcal{Q}_\epsilon(t)$ is a polynomial with nonnegative integer coefficients.

In the non-degenerate case, the above theorem becomes:

Corollary 7.4. *There exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, if the set K_ϵ of solutions of equation (1.1) is discrete and the solutions are non-degenerate, then*

$$\sum_{u \in K_\epsilon} t^{\mu(u)} = t\mathcal{P}_t(M) + t^2[\mathcal{P}_t(M) - 1] + t(1+t)\mathcal{Q}_\epsilon(t),$$

where $\mathcal{Q}_\epsilon(t)$ is a polynomial with nonnegative integer coefficients.

Since we have proved that the composition I_ϵ of ϕ_ϵ and β from M to $M_{r(M)}$ for ϵ sufficiently small is homotopic to the identity on M , the following equation holds (see [4]):

$$\mathcal{P}_t(\Sigma_{\epsilon, \delta}) = \mathcal{P}_t(M) + \mathcal{Z}(t), \quad (7.1)$$

where $\mathcal{Z}(t)$ is a polynomial with nonnegative integer coefficients (here ϵ and δ are chosen as in Proposition 5.9).

Let α and ϵ be as in Lemma 5.1, $\delta > 0$, then

$$\begin{aligned} \mathcal{P}_t \left(J_\epsilon^{m(J)+\delta}, J_\epsilon^{\frac{\alpha}{2}} \right) &= t \mathcal{P}_t(\Sigma_{\epsilon, \delta}), \\ \mathcal{P}_t \left(H_2^1(M), J_\epsilon^{m(J)+\delta} \right) &= t \left[\mathcal{P}_t \left(J_\epsilon^{m(J)+\delta}, J_\epsilon^{\frac{\alpha}{2}} \right) - t \right]. \end{aligned} \quad (7.2)$$

By Morse theory we have

$$\sum_{u \in K_\epsilon} i_t(u) = \mathcal{P}_t \left(H_2^1(M), J_\epsilon^{m(J)+\delta} \right) + \mathcal{P}_t \left(J_\epsilon^{m(J)+\delta}, J_\epsilon^{\frac{\alpha}{2}} \right) + (1+t) \mathcal{Q}_\epsilon(t),$$

where $\mathcal{Q}_\epsilon(t)$ is polynomial with nonnegative coefficients. Using this relation with (7.1) and (7.2), we obtain Theorem 7.3 and Corollary 7.4. Theorem 1.2 easily follows by evaluating the power series in $t = 1$.

Appendix

Proof of Lemma 5.2. Given any $0 < r < r(M)$, we can choose $\rho < r$ small enough so that there exists a finite open cover of M_ρ $\{C_\alpha\}_{\alpha=1, \dots, k}$ of subsets of \mathbb{R}^N with smooth charts $\xi_\alpha : D_\alpha \subset \mathbb{R}^N \rightarrow C_\alpha$ induced on M_ρ by the manifold structure of M . We assume that $D_\alpha = Z_\alpha \times T_\alpha$, with Z_α a subset of \mathbb{R}^n starshaped centred in the origin and T_α the ball of \mathbb{R}^{N-n} with centre the origin and radius ρ . For any α and any $(z, 0) \in Z_\alpha \times T_\alpha$, let $\xi_\alpha(z, 0) \in \tilde{C}_\alpha = C_\alpha \cap M$. Viceversa for any $x \in \tilde{C}_\alpha$, let $\xi_\alpha^{-1}(x) = (z, 0)$.

We denote by $\{\psi_\alpha(y)\}_{\alpha=1, \dots, k}$ a partition of unity subordinate to the cover $\{C_\alpha\}_{\alpha=1, \dots, k}$. For all $y \in M_\rho$ we write $\xi_\alpha^{-1}(y) = (z_\alpha(y), t_\alpha(y))$.

Given a function $u \in H_2^1(M)$, we define a function $v \in \mathcal{D}^{1,2}(M_r)$ by $v(y) \equiv 0$ for all $y \in M_r \setminus M_\rho$ and

$$v(y) = \sum_{\alpha=1}^k \psi_\alpha(y) u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|)$$

for all $y \in M_\rho$, where χ_ρ is defined in (2.1).

Inequality (5.4). Let us write

$$\begin{aligned} C_0 &= \left[\sup_{i, j=1, \dots, N} \sup_{\alpha=1, \dots, k} \sup_{y \in C_\alpha} (D_y(\xi_\alpha(z_\alpha(y), 0)))_{ij} \right]^2, \\ C_1 &= \left[\sup_{\substack{i=1, \dots, N, \\ j=1, \dots, N-n}} \sup_{\alpha=1, \dots, k} \sup_{y \in C_\alpha} (D(t_\alpha(y)))_{ij} \right]^2, \\ C_2 &= \sup_{\alpha=1, \dots, k} \sup_{y \in C_\alpha} (|\nabla \psi_\alpha(y)|^2 + 1), \\ C_3 &= \sup_{\alpha=1, \dots, k} \sup_{(z, t) \in D_\alpha} |\det D(\xi_\alpha(z, t))|, \\ C_4 &= \int_{\mathbb{R}^{N-n}} [(\chi_\rho(|t|))^2 + (\chi'_\rho(|t|))^2] dt. \end{aligned}$$

Then we can estimate

$$\begin{aligned}
\int_{M_r} |\nabla v(y)|^2 dy &\leq 2 \sum_{\alpha=1}^k \int_{C_\alpha} \left[|\nabla \psi_\alpha(y)|^2 (u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|))^2 \right. \\
&\quad + |\nabla_y (u(\xi_\alpha(z_\alpha(y), 0)))|^2 (\psi_\alpha(y) \chi_\rho(|t_\alpha(y)|))^2 \\
&\quad \left. + |\nabla_y (\chi_\rho(|t_\alpha(y)|))^2 (\psi_\alpha(y) u(\xi_\alpha(z_\alpha(y), 0)))^2 \right] dy \\
&\leq 2 \sum_{\alpha=1}^k \int_{C_\alpha} \left[|\nabla \psi_\alpha(y)|^2 (u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|))^2 \right. \\
&\quad + C_0 |\nabla u(\xi_\alpha(z_\alpha(y), 0))|^2 (\psi_\alpha(y) \chi_\rho(|t_\alpha(y)|))^2 \\
&\quad \left. + C_1 (\chi'_\rho(|t_\alpha(y)|))^2 (\psi_\alpha(y) u(\xi_\alpha(z_\alpha(y), 0)))^2 \right] dy \\
&\leq \sum_{\alpha=1}^k \int_{C_\alpha} \left[2C_0 |\nabla u(\xi_\alpha(z_\alpha(y), 0))|^2 (\chi_\rho(|t_\alpha(y)|))^2 \right. \\
&\quad \left. + 2(1 + C_1)C_2 (u(\xi_\alpha(z_\alpha(y), 0)))^2 [(\chi_\rho(|t_\alpha(y)|))^2 + (\chi'_\rho(|t_\alpha(y)|))^2] \right] dy \\
&\leq 2C_0 C_3 \sum_{\alpha=1}^k \int_{D_\alpha} |\nabla u(\xi_\alpha(z, 0))|^2 (\chi_\rho(|t|))^2 dz dt \\
&\quad + 2(1 + C_1)C_2 C_3 \sum_{\alpha=1}^k \int_{D_\alpha} (u(\xi_\alpha(z, 0)))^2 [(\chi_\rho(|t|))^2 + (\chi'_\rho(|t|))^2] dz dt \\
&\leq 2C_3(C_0 + (1 + C_1)C_2) \sum_{\alpha=1}^k \left[\int_{T_\alpha} (\chi_\rho(|t|))^2 dt \int_{Z_\alpha} |\nabla u(\xi_\alpha(z, 0))|^2 dz \right. \\
&\quad \left. + \int_{T_\alpha} [(\chi_\rho(|t|))^2 + (\chi'_\rho(|t|))^2] dt \int_{Z_\alpha} (u(\xi_\alpha(z, 0)))^2 dz \right] \\
&\leq 2C_3(C_0 + (1 + C_1)C_2)C_4 \sum_{\alpha=1}^k \int_{Z_\alpha} [|\nabla u(\xi_\alpha(z, 0))|^2 + (u(\xi_\alpha(z, 0)))^2] dz \\
&\leq 2C_3(C_0 + (1 + C_1)C_2)C_4 \frac{H}{h^{\frac{n}{2}}} \sum_{\alpha=1}^k \int_{\tilde{C}_\alpha} [|\nabla u(x)|_g^2 + (u(x))^2] d\mu_g.
\end{aligned}$$

One can easily see that there exists a constant $C_5 > 0$, depending only on the charts ξ_α and on the partition of unity ψ_α , such that

$$\sum_{\alpha=1}^k \int_{\tilde{C}_\alpha} [|\nabla u(x)|_g^2 + (u(x))^2] d\mu_g \leq C_5 \|u\|_{H_2^1(M)}^2$$

and by the Sobolev embedding of $H_2^1(M)$ in $L^2(M)$ (5.4) is proved.

Inequality (5.5). We show that for any $s, t \in \mathbb{R}$, $s + t \neq 0$

$$f(s + t) > \frac{c_0 \mu}{c_1} [f(s) + f(t)].$$

Let us consider first the case $|s+t| \geq 1$, $|s| \geq 1$ and $|t| \geq 1$:

$$f(s+t) \geq c_0|s+t|^p \geq c_0(|s|^p + |t|^p) \geq \frac{c_0}{c_1}(f''(s)s^2 + f''(t)t^2) > \frac{c_0\mu}{c_1}(f(s) + f(t)).$$

If $|s+t| \geq 1$, $|s| \geq 1$ and $|t| < 1$, we have:

$$f(s+t) \geq c_0(|s|^p + |t|^p) \geq c_0(|s|^p + |t|^q) > \frac{c_0\mu}{c_1}(f(s) + f(t)).$$

The same kind of inequalities holds true in the other cases.

Hereafter, for all $y \in M_r$ we denote $v_\alpha(y) = \psi_\alpha(y)u(\xi_\alpha(z_\alpha(y), 0))\chi_\rho(|t_\alpha(y)|)$. The following integrals are always meant on the intersection with the support of v :

$$\begin{aligned} \int_{M_r} f(v(y)) dy &= \int_{M_r} f\left(\sum_{\alpha=1}^k v_\alpha(y)\right) dy > \frac{c_0\mu}{c_1} \sum_{\alpha=1}^k \int_{C_\alpha} f(v_\alpha(y)) dy \\ &\geq \frac{c_0^2\mu}{c_1} \sum_{\alpha=1}^k \left[\int_{\{y \in C_\alpha \mid |v_\alpha(y)| \geq 1\}} |v_\alpha(y)|^p dy + \int_{\{y \in C_\alpha \mid |v_\alpha(y)| \leq 1\}} |v_\alpha(y)|^q dy \right] \end{aligned}$$

For all $\alpha = 1, \dots, k$ it is possible to choose $C'_\alpha \subset C_\alpha$ such that on this subset $\psi_\alpha(y) \geq \frac{1}{k}$. Then the previous chain of inequalities is bounded from below by

$$\begin{aligned} &\frac{c_0^2\mu}{c_1 k^q} \sum_{\alpha=1}^k \left[\int_{\{y \in C'_\alpha \mid |v_\alpha(y)| \geq 1\}} |u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|)|^p dy \right. \\ &\quad \left. + \int_{\{y \in C'_\alpha \mid |v_\alpha(y)| \leq 1\}} |u(\xi_\alpha(z_\alpha(y), 0)) \chi_\rho(|t_\alpha(y)|)|^q dy \right] \end{aligned} \quad (7.3)$$

Let D'_α be the set $\xi_\alpha^{-1}(C'_\alpha)$. We consider the following constants:

$$\begin{aligned} C_5 &= \inf_{\alpha=1, \dots, k} \inf_{(z, t) \in D_\alpha} |\det D(\xi_\alpha(z, t))|, \\ C_6 &= \int_{\mathbb{R}^{N-n}} (\chi_\rho(|t|))^q dt, \\ C_7 &= \inf_{\alpha=1, \dots, k} \inf_{x \in \tilde{C}_\alpha} |\det D(z_\alpha(x))|. \end{aligned}$$

The inequality (7.3) is bounded from below by

$$\begin{aligned} &\frac{c_0^2\mu C_5}{c_1 k^q} \sum_{\alpha=1}^k \left[\int_{\{(z, t) \in D'_\alpha \mid |v_\alpha(\xi_\alpha(z, t))| \geq 1\}} |u(\xi_\alpha(z, 0)) \chi_\rho(|t|)|^p dz dt \right. \\ &\quad \left. + \int_{\{(z, t) \in D'_\alpha \mid |v_\alpha(\xi_\alpha(z, t))| \leq 1\}} |u(\xi_\alpha(z, 0)) \chi_\rho(|t|)|^q dz dt \right] \end{aligned}$$

$$\begin{aligned}
&\geq \frac{c_0^2 \mu C_5}{c_1 k^q} \sum_{\alpha=1}^k \left[\int_{\{(z,t) \in D'_\alpha \mid |u(\xi_\alpha(z,0))| \geq 1\}} |u(\xi_\alpha(z,0))|^p (\chi_\rho(|t|))^q dz dt \right. \\
&\quad + \int_{\{(z,t) \in D'_\alpha \mid |u(\xi_\alpha(z,0))| \leq 1\}} |u(\xi_\alpha(z,0))|^q (\chi_\rho(|t|))^q dz dt \\
&\quad - \int_{\{(z,t) \in D'_\alpha \mid |v_\alpha(\xi_\alpha(z,t))| \leq 1, |u(\xi_\alpha(z,0))| \geq 1\}} |u(\xi_\alpha(z,0))|^p (\chi_\rho(|t|))^q dz dt \\
&\quad \left. + \int_{\{(z,t) \in D'_\alpha \mid |v_\alpha(\xi_\alpha(z,t))| \leq 1, |u(\xi_\alpha(z,0))| \geq 1\}} |u(\xi_\alpha(z,0))|^q (\chi_\rho(|t|))^q dz dt \right] \\
&= \frac{c_0^2 \mu C_5 C_6}{c_1 k^q} \sum_{\alpha=1}^k \left[\int_{\{(z,0) \in D'_\alpha \mid |u(\xi_\alpha(z,0))| \geq 1\}} |u(\xi_\alpha(z,0))|^p dz \right. \\
&\quad \left. + \int_{\{(z,0) \in D'_\alpha \mid |u(\xi_\alpha(z,0))| \leq 1\}} |u(\xi_\alpha(z,0))|^q dz \right] \\
&\geq \frac{c_0^2 \mu C_5 C_6 C_7}{c_1 k^q} \sum_{\alpha=1}^k \left[\int_{\{x \in \tilde{C}_\alpha \mid x \in C'_\alpha, |u(x)| \geq 1\}} |u(x)|^p dx \right. \\
&\quad \left. + \int_{\{x \in \tilde{C}_\alpha \mid x \in C'_\alpha, |u(x)| \leq 1\}} |u(x)|^q dx \right].
\end{aligned}$$

Since for all $x \in M$ the sum of the $\psi_\alpha(x)$ is one, there exists $\hat{\alpha}$ such that $x \in C'_{\hat{\alpha}}$. Then for any $u \in L^1(M)$

$$\begin{aligned}
\sum_{\alpha=1}^k \int_{C'_\alpha \cap M} |u(x)| dx &= \sum_{\alpha=1}^k \int_M \chi_{C'_\alpha}(x) |u(x)| dx = \int_M \left(\sum_{\alpha=1}^k \chi_{C'_\alpha}(x) \right) |u(x)| dx \\
&\geq \int_M |u(x)| dx.
\end{aligned}$$

This means that

$$\begin{aligned}
&\sum_{\alpha=1}^k \left[\int_{\{x \in \tilde{C}_\alpha \mid x \in C'_\alpha, |u(x)| \geq 1\}} |u(x)|^p dx + \int_{\{x \in \tilde{C}_\alpha \mid x \in C'_\alpha, |u(x)| \leq 1\}} |u(x)|^q dx \right] \\
&\geq \int_{\{x \in M \mid |u(x)| \geq 1\}} |u(x)|^p dx + \int_{\{x \in M \mid |u(x)| \leq 1\}} |u(x)|^q dx \\
&\geq \frac{1}{c_1} \int_M f''(u(x))(u(x))^2 dx > \frac{\mu}{c_1} \int_M f(u(x)) dx \geq \frac{\mu}{c_1 H^{\frac{n}{2}}} \int_M f(u(x)) d\mu_g.
\end{aligned}$$

Inequality (5.6). For $s > 0$ $f(s)$ is increasing. Then we have

$$\begin{aligned}
\int_{M_r} f(v(y)) dy &< \frac{c_1}{c_0\mu} \int_{M_r} f(|v(y)|) dy \leq \frac{c_1}{c_0\mu} \int_{M_r} f\left(\sum_{\alpha=1}^k |v_\alpha(y)|\right) dy \\
&\leq \frac{c_1}{c_0\mu} \int_{M_r} f\left(\sum_{\alpha=1}^k |\psi_\alpha(y)u(\xi_\alpha(z_\alpha(y), 0))|\right) dy \\
&= \frac{c_1}{c_0\mu} \sum_{\beta=1}^k \int_{C_\beta} \psi_\beta(y) f\left(\sum_{\alpha=1}^k |\psi_\alpha(y)u(\xi_\alpha(z_\alpha(y), 0))|\right) dy \\
&\leq \frac{c_1 C_3}{c_0\mu} \sum_{\beta=1}^k \int_{D_\beta} f\left(\sum_{\alpha=1}^k |\chi_{D_\alpha}(z, t)u(\xi_\alpha(z, 0))|\right) dz dt \\
&\leq \frac{c_1 C_3 C_8}{c_0\mu} \sum_{\beta=1}^k \int_{Z_\beta} f\left(\sum_{\alpha=1}^k |\chi_{Z_\alpha}(z)u(\xi_\alpha(z, 0))|\right) dz,
\end{aligned}$$

where C_8 is the volume of the ball of radius ρ in \mathbb{R}^{N-n} . Proceeding with the chain of inequalities we obtain

$$\begin{aligned}
\sum_{\beta=1}^k \int_{Z_\beta} f\left(\sum_{\alpha=1}^k |\chi_{Z_\alpha}(z)u(\xi_\alpha(z, 0))|\right) dz &= \sum_{\beta=1}^k \int_{\tilde{C}_\beta} f\left(\sum_{\alpha=1}^k |\chi_{\tilde{C}_\alpha}(x)u(x)|\right) dx \\
&\leq k \int_M f(k|u(x)|) dx \\
&< \frac{k c_1}{\mu} \left[\int_{\{x \in M \mid k|u(x)| \geq 1\}} k^p |u(x)|^p dx + \int_{\{x \in M \mid k|u(x)| \leq 1\}} k^q |u(x)|^q dx \right] \\
&= \frac{k c_1}{\mu} \left[\int_{\{x \in M \mid |u(x)| \geq 1\}} k^p |u(x)|^p dx + \int_{\{x \in M \mid |u(x)| \leq 1\}} k^q |u(x)|^q dx \right. \\
&\quad \left. + \int_{\{x \in M \mid |u(x)| \leq 1, k|u(x)| \geq 1\}} k^p |u(x)|^p dx - \int_{\{x \in M \mid |u(x)| \leq 1, k|u(x)| \geq 1\}} k^q |u(x)|^q dx \right] \\
&\leq \frac{k c_1}{\mu} \left[\int_{\{x \in M \mid |u(x)| \geq 1\}} k^p |u(x)|^p dx + \int_{\{x \in M \mid |u(x)| \leq 1\}} k^q |u(x)|^q dx \right] \\
&\leq \frac{k^{q+1} c_1}{c_0\mu} \int_M f(u(x)) dx \leq \frac{k^{q+1} c_1}{c_0\mu h^{\frac{n}{2}}} \int_M f(u(x)) d\mu_g.
\end{aligned}$$

Inequality (5.7). The proof is analogous to the proof of (5.5). □

We complete now the proof of Proposition 5.4.

Proof of equation (5.20). The following inequalities hold:

$$\begin{aligned}
& \frac{1}{\epsilon_k^n} \int_B |f''(u_k^*)u_k^* - f'(u_k^*)|^{\frac{2n}{n+2}} d\mu_g \\
& \leq \frac{2^{\frac{2n}{n+2}}}{\epsilon_k^n} \int_B \left(|f''(u_k^*)u_k^*|^{\frac{2n}{n+2}} + |f'(u_k^*)|^{\frac{2n}{n+2}} \right) d\mu_g \\
& < \frac{2(2c_1)^{\frac{2n}{n+2}}}{\epsilon_k^n} \left(\int_{\{x \in B \mid |u_k^*(x)| \geq 1\}} |u_k^*(x)|^{\frac{(p-1)2n}{n+2}} d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1\}} |u_k^*(x)|^{\frac{(q-1)2n}{n+2}} d\mu_g \right) \\
& \leq \frac{2(2c_1)^{\frac{2n}{n+2}}}{\epsilon_k^n} \left(\int_{\{x \in B \mid |u_k^*(x)| \geq 1\}} |u_k^*(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1\}} |u_k^*(x)|^q d\mu_g \right),
\end{aligned}$$

where in the last inequality we have used the fact that $\frac{(p-1)2n}{n+2} < p$ and $\frac{(q-1)2n}{n+2} > q$. We can write

$$\begin{aligned}
& \int_{\{x \in B \mid |u_k^*(x)| \geq 1\}} |u_k^*(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1\}} |u_k^*(x)|^q d\mu_g \\
& = \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \geq 1\}} |u_k^*(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \leq 1\}} |u_k^*(x)|^p d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \leq 1\}} |u_k^*(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \geq 1\}} |u_k^*(x)|^p d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \geq 1\}} |u_k^*(x)|^q d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \leq 1\}} |u_k^*(x)|^q d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \leq 1\}} |u_k^*(x)|^q d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \geq 1\}} |u_k^*(x)|^q d\mu_g \\
& \leq \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \geq 1\}} |u_k^*(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \leq 1\}} |u_k^*(x)|^q d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \leq 1\}} |u_k^*(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \geq 1\}} |u_k^*(x)|^p d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \geq 1\}} |u_k^*(x)|^q d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \leq 1\}} |u_k^*(x)|^q d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \leq 1\}} |u_k^*(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \geq 1\}} |u_k^*(x)|^p d\mu_g \\
& \leq \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \geq 1\}} 2^p (|\tilde{u}_k(x)|^p + |u_k(x)|^p) d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \leq 1\}} 2^q (|\tilde{u}_k(x)|^q + |u_k(x)|^q) d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \leq 1\}} 2^p |\tilde{u}_k(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \geq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \geq 1\}} 2^p |u_k(x)|^p d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \geq 1\}} 2^p (|\tilde{u}_k(x)|^p + |u_k(x)|^p) d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \leq 1\}} 2^q (|\tilde{u}_k(x)|^q + |u_k(x)|^q) d\mu_g \\
& \quad + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \geq 1, |u_k(x)| \leq 1\}} 2^p |\tilde{u}_k(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k^*(x)| \leq 1, |\tilde{u}_k(x)| \leq 1, |u_k(x)| \geq 1\}} 2^p |u_k(x)|^p d\mu_g
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\{x \in B \mid |\tilde{u}_k(x)| \geq 1\}} 2^p |\tilde{u}_k(x)|^p d\mu_g + \int_{\{x \in B \mid |\tilde{u}_k(x)| \leq 1\}} 2^q |\tilde{u}_k(x)|^q d\mu_g + \int_{\{x \in B \mid |u_k(x)| \geq 1\}} 2^p |u_k(x)|^p d\mu_g + \int_{\{x \in B \mid |u_k(x)| \leq 1\}} 2^q |u_k(x)|^q d\mu_g \\
&\leq \frac{2^q}{c_0} \int_M [f(\tilde{u}_k) + f(u_k)] d\mu_g.
\end{aligned}$$

Concluding there exists a constant $C > 0$ such that

$$\begin{aligned}
\frac{1}{\epsilon_k^n} \int_B |f''(u_k^*) u_k^* - f'(u_k^*)|^{\frac{2n}{n+2}} d\mu_g &< \frac{C}{\epsilon_k^n} \int_M [f(\tilde{u}_k) + f(u_k)] d\mu_g \\
&\leq \frac{2C}{(\mu - 2)} [J_{\epsilon_k}(\tilde{u}_k) + J_{\epsilon_k}(u_k)] \leq \frac{8Cm(J)}{(\mu - 2)}
\end{aligned}$$

and this completes the proof of (5.20). \square

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